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Spectral Theory of Operators on Hilbert Spaces

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To *A* & *J* & *A* & *A* & *B*

... and all life long my instinct has been to abandon anything for which I have no talent; tennis, golf, dancing, sallying, all have been abandoned, and perhaps it is desperation which keeps me writing...

Graham Greene

Preface

This work is an introduction to the spectral theory of Hilbert space operators. My main goal is to offer a modern introductory textbook for a first course in spectral theory for graduate students, emphasizing recent aspects of the theory, with detailed proofs. The book is addressed to a wide audience consisting of graduate students in mathematics, as well as those studying statistics, economics, engineering, and physics. The text, however, can also be useful to working mathematicians delving into the spectral theory of Hilbert space operators, and for scientists wishing to apply spectral theory to their field.

The prerequisite for this book is a first course in elementary functional analysis including, in particular, operator theory on Hilbert spaces at an introductory level. Of course, this includes a formal introduction to analysis, covering elements of measure theory and functions of a complex variable. However, every effort has been made to ensure that this book, despite its relative shortness, is as self-contained as possible.

Chapter 1 summarizes the basic concepts from Hilbert space theory that will be required in the text. By no means is it intended to replace a formal introductory course on single operator theory. Its main purpose, besides summarizing the basics required for further chapters, is to unify notation and terminology. Chapter 2 discusses standard spectral results for (bounded linear) operators on Banach and Hilbert spaces, including the classical partition of the spectrum and spectral properties for specific classes of operators. Chapter 3 is devoted wholly to the Spectral Theorem for normal operators. After considering the compact case, it meticulously treats the general case, carrying out the proofs of both versions of the Spectral Theorem for the general case in great detail. The idea behind those highly detailed proofs is to discuss and explain delicate points, stressing some usually hidden features. The chapter closes with the Fuglede Theorems. Chapter 4 deals with functional calculus for normal operators, which depends on the Spectral Theorem, and also with analytic functional calculus (i.e., Riesz functional calculus). The same level of detail employed in the proofs of Chapter 3 is repeated here. Chapter 5 focuses

on Fredholm theory and compact perturbations of the spectrum, where a finer analysis of the spectrum is worked out, leading to further partitions involving the essential spectrum, the Weyl spectrum, and the Browder spectrum. The chapter closes with a discussion of Weyl's and Browder's Theorems, including some very recent results.

The final section of each chapter is a section on *Additional Propositions*, consisting of either auxiliary results that will be required to support a proof in the main text, or further results extending some theorems proved in the main text. These are followed by a set of *Notes*, where each proposition is briefly discussed, and references are provided indicating proofs for all of them. The Additional Propositions can also be thought of as a set of proposed problems, and their respective Notes can be viewed as hints for solving them.

At the end of each chapter, the reader will find a collection of suggested readings. This has a triple purpose: to offer a reasonable bibliography including most of the classics as well as some quite recent texts; to indicate where different approaches and proofs can be found; and to indicate where further results can be found. In this sense, some of the references are suggested as a second reading on the subject.

The material in this book has been prepared to be covered in a one-semester graduate course. The resulting text is the outcome of attempts to meet the needs of a contemporary first course in spectral theory for an audience as described in the first paragraph, possessing the prerequisites listed in the second paragraph of this Preface. The logical dependence of the various sections (and chapters) is roughly linear and reflects approximately the minimum amount of material needed to proceed further.

Rio de Janeiro, March 2012   Carlos S. Kubrusly

Contents

1

Preliminaries

This chapter summarizes the background required for the rest of the book. Its purpose is threefold: notation, terminology, and basic results. By basic results we mean well-known theorems from single operator theory that will be needed in the sequel. As usual, the set of nonnegative integers, positive integers, and all integers will be denoted by $\mathbb{N}_0$, $\mathbb{N}$, $\mathbb{Z}$, and the set of rational numbers, real numbers, and complex numbers will denoted by $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$, respectively.

1.1 Notation and Terminology

We assume the reader is familiar with the notions of linear space (or vector space), normed space, Banach space, inner product space, and Hilbert space. All spaces considered in this book are complex (i.e., over the complex field $\mathbb{C}$). Given a (complex) inner product space $\mathcal{X}$, the sesquilinear form (linear in the first argument) $\langle\,\cdot\,;\cdot\,\rangle\colon \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ stands for the inner product in $\mathcal{X}$. We do not distinguish notation for norms. Thus $\|\cdot\|$ denotes the norm on a normed space $\mathcal{X}$ (in particular, the norm generated by the inner product in an inner product space $\mathcal{X}$ — i.e., $\|x\|^2 = \langle x\,;x\rangle$ for all x in $\mathcal{X}$) and also the (induced uniform) operator norm in $\mathcal{B}[\mathcal{X},\mathcal{Y}]$ (i.e., $\|T\| = \sup_{x\neq 0} \frac{\|Tx\|}{\|x\|}$ for all T in $\mathcal{B}[\mathcal{X},\mathcal{Y}]$), where $\mathcal{B}[\mathcal{X},\mathcal{Y}]$ stands for the normed space of all bounded linear transformations of a normed space $\mathcal{X}$ into a normed space $\mathcal{Y}$. The induced uniform norm has the *operator norm property*, which means that if $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ are normed spaces over the same scalar field, and if $T \in \mathcal{B}[\mathcal{X},\mathcal{Y}]$ and $S \in \mathcal{B}[\mathcal{Y},\mathcal{Z}]$, then $ST \in \mathcal{B}[\mathcal{X},\mathcal{Z}]$ and $\|ST\| \leq \|S\|\|T\|$.

Between normed spaces, *continuous linear transformation* and *bounded linear transformation* are synonyms. A transformation $T\colon \mathcal{X} \to \mathcal{Y}$ between normed spaces $\mathcal{X}$ and $\mathcal{Y}$ is *bounded* if there exists a constant $\beta \geq 0$ such that $\|Tx\| \leq \beta\|x\|$ for every x in $\mathcal{X}$. It is said to be *bounded below* if there is a constant $\alpha > 0$ such that $\alpha\|x\| \leq \|Tx\|$ for every x in $\mathcal{X}$. An *operator* on a normed space $\mathcal{X}$ is precisely a *bounded linear* (i.e., a continuous linear) transformation of $\mathcal{X}$ into itself. Set $\mathcal{B}[\mathcal{X}] = \mathcal{B}[\mathcal{X},\mathcal{X}]$ for short: the normed algebra of all operators on $\mathcal{X}$. If $\mathcal{X} \neq \{0\}$, then $\mathcal{B}[\mathcal{X}]$ contains the identity operator I and $\|I\| = 1$ so that $\mathcal{B}[\mathcal{X}]$ is a *unital normed algebra*. Moreover, if $\mathcal{X} \neq \{0\}$

is a Banach space, then $\mathcal{B}[\mathcal{X}]$ is a *unital (complex) Banach algebra*. Since the induced uniform norm has the operator norm property, a trivial induction shows that $\|T^n\| \leq \|T\|^n$, for every operator $T \in \mathcal{B}[\mathcal{X}]$ on a normed space $\mathcal{X}$, for each integer $n \geq 0$.

A transformation $T \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$ is a *contraction* if $\|T\| \leq 1$; equivalently, if $\|Tx\| \leq \|x\|$ for every $x \in \mathcal{X}$. It is a *strict contraction* if $\|T\| < 1$. If $\mathcal{X} \neq \{0\}$, then a transformation T is a contraction (or a strict contraction) if and only if $\sup_{x \neq 0}(\|Tx\|/\|x\|) \leq 1$ (or $\sup_{x \neq 0}(\|Tx\|/\|x\|) < 1$). If $\mathcal{X} \neq \{0\}$, then

$$\|T\| < 1 \quad \Longrightarrow \quad \|Tx\| < \|x\| \ \text{ for every } \ 0 \neq x \in \mathcal{X} \quad \Longrightarrow \quad \|T\| \leq 1.$$

If T satisfies the middle inequality, then it is called a *proper contraction*.

Let $\mathcal{X}$ be a Banach space. A sequence $\{T_n\}$ of operators in $\mathcal{B}[\mathcal{X}]$ *converges uniformly* to an operator T in $\mathcal{B}[\mathcal{X}]$ if $\|T_n - T\| \to 0$, and $\{T_n\}$ *converges strongly* to T if $\|(T_n - T)x\| \to 0$ for every x in $\mathcal{X}$. If $\mathcal{X}$ is a Hilbert space, then $\{T_n\}$ *converges weakly* to T if $\langle(T_n - T)x \, ; y\rangle \to 0$ for every x, y in $\mathcal{X}$ (or, equivalently, $\langle(T_n - T)x \, ; x\rangle \to 0$ for every x in $\mathcal{X}$, if the Hilbert space is complex). These will be denoted by $T_n \xrightarrow{u} T$, $T_n \xrightarrow{s} T$, or $T_n \xrightarrow{w} T$, respectively. The sequence $\{T_n\}$ is *bounded* if $\sup_n \|T_n\| < \infty$. It is readily verified that

$$T_n \xrightarrow{u} T \quad \Longrightarrow \quad T_n \xrightarrow{s} T \quad \Longrightarrow \quad T_n \xrightarrow{w} T \quad \Longrightarrow \quad \sup_n \|T_n\| < \infty$$

(the last implication is a consequence of the Banach–Steinhaus Theorem).

By a *subspace* of a normed space we mean a *closed* linear manifold of it. The closure of a linear manifold of a normed space is a subspace. A subspace of a Banach space is a Banach space. A subspace $\mathcal{M}$ of $\mathcal{X}$ is *nontrivial* if $\{0\} \neq \mathcal{M} \neq \mathcal{X}$. Let $T \in \mathcal{B}[\mathcal{X}]$ be an operator on a normed space $\mathcal{X}$. A subset A of $\mathcal{X}$ is *T-invariant* (or A is an *invariant subset* for T) if $T(A) \subseteq A$ (i.e., if $Tx \in A$ whenever $x \in A$). An *invariant linear manifold* (*invariant subspace*) for T is a linear manifold (subspace) of $\mathcal{X}$ that, as a subset of $\mathcal{X}$, is T-invariant. The zero space $\{0\}$ and the whole space $\mathcal{X}$ (i.e., the trivial subspaces) are trivially invariant for every T in $\mathcal{B}[\mathcal{X}]$. If $\mathcal{M}$ is an invariant linear manifold for T, then its closure $\mathcal{M}^-$ is an invariant subspace for T.

Let $\mathcal{X}$ and $\mathcal{Y}$ be linear spaces. The *kernel* (or *null space*) of a linear transformation $T \colon \mathcal{X} \to \mathcal{Y}$ is the inverse image of $\{0\}$ under T,

$$\mathcal{N}(T) = T^{-1}(\{0\}) = \big\{x \in \mathcal{X} \colon Tx = 0\big\},$$

which is a linear manifold of $\mathcal{X}$. The *range* of T is the image of $\mathcal{X}$ under T,

$$\mathcal{R}(T) = T(\mathcal{X}) = \big\{y \in \mathcal{Y} \colon y = Tx \text{ for some } x \in \mathcal{X}\big\},$$

which is a linear manifold of $\mathcal{Y}$. If $\mathcal{X}$ and $\mathcal{Y}$ are normed spaces and T lies in $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$ (i.e., if T is bounded), then $\mathcal{N}(T)$ is a subspace of $\mathcal{X}$ (i.e., if T is bounded, then $\mathcal{N}(T)$ is closed).

If $\mathcal{M}$ and $\mathcal{N}$ are linear manifolds of a linear space $\mathcal{X}$, then the *ordinary sum* (or simply, the *sum*) $\mathcal{M} + \mathcal{N}$ is the linear manifold of $\mathcal{X}$ consisting of all sums $u + v$ with u in $\mathcal{M}$ and v in $\mathcal{N}$. The *direct sum* $\mathcal{M} \oplus \mathcal{N}$ is the linear space of all ordered pairs (u, v) with u in $\mathcal{M}$ and v in $\mathcal{N}$, where vector addition and scalar multiplication are defined coordinatewise. Although ordinary and direct sums are different linear spaces, they are isomorphic if $\mathcal{M} \cap \mathcal{N} = \{0\}$. A pair of linear manifolds $\mathcal{M}$ and $\mathcal{N}$ are *algebraic complements* of each other if $\mathcal{M} + \mathcal{N} = \mathcal{X}$ and $\mathcal{M} \cap \mathcal{N} = \{0\}$. In this case, each vector x in $\mathcal{X}$ can be uniquely written as $x = u + v$ with u in $\mathcal{M}$ and v in $\mathcal{N}$. If two subspaces (linear manifolds) of a normed space are algebraic complements of each other, then they are called *complementary subspaces* (*complementary linear manifolds*).

1.2 Inverse Theorems

One of the fundamental theorems of (linear) functional analysis, the OPEN MAPPING THEOREM, says that *a continuous linear transformation of a Banach space onto a Banach space is an open mapping* (see, e.g., [66, Section 4.5] — a mapping is *open* if it maps open sets onto open sets). A crucial corollary of it is the following result.

Theorem 1.1. (INVERSE MAPPING THEOREM). *If $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces and if $T \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$ is injective and surjective, then $T^{-1} \in \mathcal{B}[\mathcal{Y}, \mathcal{X}]$.*

Proof. An invertible transformation is precisely an injective and surjective one. Since the inverse of an invertible linear transformation is linear, and since an invertible transformation is open if and only if it has a continuous inverse, the stated result follows from the Open Mapping Theorem. $\square$

Thus an injective and surjective bounded linear transformation between Banach spaces has a *bounded* (linear) inverse. Let $\mathcal{G}[\mathcal{X}, \mathcal{Y}]$ denote the collection of all invertible (i.e., injective and surjective) elements from $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$ with a bounded inverse. (Recall that the inverse of any linear transformation is linear.) The above theorem says that, if $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, then every invertible transformation from $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$ lies in $\mathcal{G}[\mathcal{X}, \mathcal{Y}]$. Set $\mathcal{G}[\mathcal{X}] = \mathcal{G}[\mathcal{X}, \mathcal{X}]$, which forms a group under multiplication whenever $\mathcal{X}$ is a Banach space, viz., the group of all invertible operators from $\mathcal{B}[\mathcal{X}]$ (every operator in $\mathcal{G}[\mathcal{X}]$ has an inverse in $\mathcal{G}[\mathcal{X}]$). Here is a useful corollary of the Inverse Mapping Theorem.

Theorem 1.2. (BOUNDED INVERSE THEOREM). *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and take any $T \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$. The following assertions are equivalent.*

(a) *T has a bounded inverse on its range.*

(b) *T is bounded below.*

(c) *T is injective and has a closed range.*

Proof. Part (i). The equivalence between (a) and (b) still holds if $\mathcal{X}$ and $\mathcal{Y}$ are just normed spaces. Indeed, if there exists $T^{-1} \in \mathcal{B}[\mathcal{R}(T), \mathcal{X}]$, then there exists a constant $\beta > 0$ such that $\|T^{-1}y\| \leq \beta \|y\|$ for every $y \in \mathcal{R}(T)$. Take an arbitrary $x \in \mathcal{X}$ so that $Tx \in \mathcal{R}(T)$. Thus $\|x\| = \|T^{-1}Tx\| \leq \beta \|Tx\|$, and so $\frac{1}{\beta}\|x\| \leq \|Tx\|$. Hence (a) implies (b). Conversely, if (b) holds true, then $0 < \|Tx\|$ for every nonzero x in $\mathcal{X}$, and so $\mathcal{N}(T) = \{0\}$, which means that T has a (linear) inverse on its range — *a linear transformation is injective if and only if it has a null kernel*. Take an arbitrary $y \in \mathcal{R}(T)$ so that $y = Tx$ for some $x \in \mathcal{X}$. Thus $\|T^{-1}y\| = \|T^{-1}Tx\| = \|x\| \leq \frac{1}{\alpha}\|Tx\| = \frac{1}{\alpha}\|y\|$, which means that T^{-1} is bounded. Therefore (b) implies (a).

Part (ii). Take an arbitrary $\mathcal{R}(T)$-valued convergent sequence $\{y_n\}$. Since each y_n lies in $\mathcal{R}(T)$, there exists an $\mathcal{X}$-valued sequence $\{x_n\}$ such that $y_n = Tx_n$ for each n. Since $\{Tx_n\}$ converges in $\mathcal{Y}$, it is a Cauchy sequence in $\mathcal{Y}$. Thus, if T is bounded below, then there exists $\alpha > 0$ such that

$$0 \leq \alpha \|x_m - x_n\| \leq \|T(x_m - x_n)\| = \|Tx_m - Tx_n\|$$

for every m, n so that $\{x_n\}$ is a Cauchy sequence in $\mathcal{X}$, and thus it converges in $\mathcal{X}$ to, say, $x \in \mathcal{X}$ if $\mathcal{X}$ is a Banach space. Since T is continuous, it preserves convergence so that $y_n = Tx_n \to Tx$, which implies that the (unique) limit of $\{y_n\}$ lies in $\mathcal{R}(T)$. Conclusion: $\mathcal{R}(T)$ is closed in $\mathcal{Y}$ by the classical Closed Set Theorem. That is, $\mathcal{R}(T)^- = \mathcal{R}(T)$ whenever $\mathcal{X}$ is a Banach space, where $\mathcal{R}(T)^-$ stands for the closure of $\mathcal{R}(T)$. Moreover, since (b) trivially implies $\mathcal{N}(T) = \{0\}$, it follows that (b) implies (c). On the other hand, if $\mathcal{N}(T) = \{0\}$, then T is injective. If, in addition, the linear manifold $\mathcal{R}(T)$ is closed in the Banach space $\mathcal{Y}$, then it is itself a Banach space so that $T: \mathcal{X} \to \mathcal{R}(T)$ is an injective and surjective bounded linear transformation of the Banach space $\mathcal{X}$ onto the Banach space $\mathcal{R}(T)$. Therefore, its inverse T^{-1} lies in $\mathcal{B}[\mathcal{R}(T), \mathcal{X}]$ by the Inverse Mapping Theorem. That is, (c) implies (a). $\qquad\square$

Notice that if one of the Banach spaces $\mathcal{X}$ or $\mathcal{Y}$ is zero, then the results in the previous theorems are either void or hold trivially. Indeed, if $\mathcal{X} \neq \{0\}$ and $\mathcal{Y} = \{0\}$, then the unique operator in $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$ is not injective (thus not bounded below and not invertible); if $\mathcal{X} = \{0\}$ and $\mathcal{Y} \neq \{0\}$, then the unique operator in $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$ is not surjective but is bounded below (thus injective) and has a bounded inverse on its singleton range; if $\mathcal{X} = \{0\}$ and $\mathcal{Y} = \{0\}$ the unique operator in $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$ has an inverse in $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$.

The next theorem is a rather useful result establishing a *power series expansion* for $(\lambda I - T)^{-1}$. This is the *Neumann expansion* (or *Neumann series*) due to C.G. Neumann. The condition $\|T\| < |\lambda|$ will be weakened in Corollary 2.12.

Theorem 1.3. (Neumann Expansion). *If $T \in \mathcal{B}[\mathcal{X}]$ is an operator on a Banach space $\mathcal{X}$, and if λ is any scalar such that $\|T\| < |\lambda|$, then $\lambda I - T$ has an inverse in $\mathcal{B}[\mathcal{X}]$ given by the following uniformly convergent series:*

$$(\lambda I - T)^{-1} = \frac{1}{\lambda} \sum_{k=0}^{\infty} \left(\frac{T}{\lambda}\right)^k.$$

Proof. Take an arbitrary operator $T \in \mathcal{B}[\mathcal{X}]$ and an arbitrary nonzero scalar λ. If $\|T\| < |\lambda|$, then $\sup_n \sum_{k=0}^{n} \left\|\left(\frac{T}{\lambda}\right)\right\|^k < \infty$. Thus, since $\sum_{k=0}^{n} \left\|\left(\frac{T}{\lambda}\right)^k\right\| \leq \sum_{k=0}^{n} \left\|\left(\frac{T}{\lambda}\right)\right\|^k$ for every $n \geq 0$, it follows that the increasing sequence of non-negative numbers $\left\{\sum_{k=0}^{n} \left\|\left(\frac{T}{\lambda}\right)^k\right\|\right\}$ is bounded, and hence it converges in $\mathbb{R}$, which means that the sequence $\left\{\left(\frac{T}{\lambda}\right)^n\right\}$ is absolutely summable, and so it is summable (since $\mathcal{B}[\mathcal{X}]$ is a Banach space — see, e.g., [66, Proposition 4.4]). That is, the series $\sum_{k=0}^{\infty} \left(\frac{T}{\lambda}\right)^k$ converges in $\mathcal{B}[\mathcal{X}]$. Equivalently, there is an operator in $\mathcal{B}[\mathcal{X}]$, say $\sum_{k=0}^{\infty} \left(\frac{T}{\lambda}\right)^k$, such that

$$\sum\nolimits_{k=0}^{n} \left(\tfrac{T}{\lambda}\right)^k \xrightarrow{u} \sum\nolimits_{k=0}^{\infty} \left(\tfrac{T}{\lambda}\right)^k.$$

Now it is also readily verified by induction that, for every $n \geq 0$,

$$(\lambda I - T)\tfrac{1}{\lambda}\sum\nolimits_{k=0}^{n} \left(\tfrac{T}{\lambda}\right)^k = \tfrac{1}{\lambda}\sum\nolimits_{k=0}^{n} \left(\tfrac{T}{\lambda}\right)^k (\lambda I - T) = I - \left(\tfrac{T}{\lambda}\right)^{n+1}.$$

But $\left(\frac{T}{\lambda}\right)^n \xrightarrow{u} O$ (since $\left\|\left(\frac{T}{\lambda}\right)^n\right\| \leq \left\|\left(\frac{T}{\lambda}\right)\right\|^n \to 0$ when $\|T\| < |\lambda|$), and so

$$(\lambda I - T)\tfrac{1}{\lambda}\sum\nolimits_{k=0}^{n} \left(\tfrac{T}{\lambda}\right)^k \xrightarrow{u} I \quad \text{and} \quad \tfrac{1}{\lambda}\sum\nolimits_{k=0}^{n} \left(\tfrac{T}{\lambda}\right)^k (\lambda I - T) \xrightarrow{u} I.$$

Hence

$$(\lambda I - T)\tfrac{1}{\lambda}\sum\nolimits_{k=0}^{\infty} \left(\tfrac{T}{\lambda}\right)^k = \tfrac{1}{\lambda}\sum\nolimits_{i=0}^{\infty} \left(\tfrac{T}{\lambda}\right)^k (\lambda I - T) = I,$$

and so $\frac{1}{\lambda} \sum_{k=0}^{\infty} \left(\frac{T}{\lambda}\right)^k \in \mathcal{B}[\mathcal{X}]$ is the inverse of $\lambda I - T \in \mathcal{B}[\mathcal{X}]$. $\square$

Remark. Observe that, under the hypothesis of Theorem 1.3,

$$\|(\lambda I - T)^{-1}\| \leq \tfrac{1}{|\lambda|}\sum\nolimits_{k=0}^{\infty} \left(\tfrac{\|T\|}{|\lambda|}\right)^k = \left(|\lambda| - \|T\|\right)^{-1}.$$

1.3 Orthogonal Structure

Two vectors x and y in an inner product space $\mathcal{X}$ are *orthogonal* if $\langle x \, ; y \rangle = 0$. In this case we write $x \perp y$. Two subsets A and B of $\mathcal{X}$ are *orthogonal* (notation: $A \perp B$) if every vector in A is orthogonal to every vector in B. The *orthogonal complement* of a set A is the set $A^\perp$ made up of all vectors in $\mathcal{X}$ that are orthogonal to every vector of A,

$$A^\perp = \left\{x \in \mathcal{X} \colon \langle x \, ; y \rangle = 0 \text{ for every } y \in A\right\},$$

which is a subspace (i.e., a closed linear manifold) of $\mathcal{X}$, and therefore $A^\perp = (A^\perp)^- = (A^-)^\perp$. If $\mathcal{M}$ and $\mathcal{N}$ are orthogonal ($\mathcal{M} \perp \mathcal{N}$) linear manifolds of an inner product space $\mathcal{X}$, then $\mathcal{M} \cap \mathcal{N} = \{0\}$. In particular, $\mathcal{M} \cap \mathcal{M}^\perp = \{0\}$. If $\mathcal{M} \perp \mathcal{N}$, then $\mathcal{M} \oplus \mathcal{N}$ is an *orthogonal direct sum*. If $\mathcal{X}$ is a Hilbert space, then $\mathcal{M}^{\perp\perp} = \mathcal{M}^-$, and $\mathcal{M}^\perp = \{0\}$ if and only if $\mathcal{M}^- = \mathcal{X}$. Moreover, *if $\mathcal{M}$ and $\mathcal{N}$ are orthogonal subspaces of a Hilbert space $\mathcal{X}$, then $\mathcal{M} + \mathcal{N}$ is a subspace*

of $\mathcal{X}$. In this case, $\mathcal{M} + \mathcal{N} \cong \mathcal{M} \oplus \mathcal{N}$: ordinary and direct sums of orthogonal subspaces of a Hilbert space are unitarily equivalent — the inner product in $\mathcal{M} \oplus \mathcal{N}$ is given by $\langle (u_1, v_1) ; (u_2, v_2) \rangle = \langle u_1 ; u_2 \rangle + \langle v_1 ; v_2 \rangle$ for every (u_1, v_1) and (u_2, v_2) in $\mathcal{M} \oplus \mathcal{N}$. Two linear manifolds $\mathcal{M}$ and $\mathcal{N}$ are *orthogonal complementary* if $\mathcal{M} + \mathcal{N} = \mathcal{X}$ and $\mathcal{M} \perp \mathcal{N}$. If $\mathcal{M}$ and $\mathcal{N}$ are orthogonal complementary subspaces of a Hilbert space $\mathcal{X}$, then we identify the ordinary sum $\mathcal{M} + \mathcal{N}$ with the orthogonal sum $\mathcal{M} \oplus \mathcal{N}$, and write (an abuse of notation, actually) $\mathcal{X} = \mathcal{M} \oplus \mathcal{N}$ (instead of $\mathcal{X} \cong \mathcal{M} \oplus \mathcal{N}$).

A central result of Hilbert space geometry, the PROJECTION THEOREM, says that, *if $\mathcal{H}$ is a Hilbert space, and $\mathcal{M}$ is any subspace of $\mathcal{H}$, then*

$$\mathcal{H} = \mathcal{M} + \mathcal{M}^\perp.$$

Ordinary and direct sums of *orthogonal* subspaces are unitarily equivalent. Thus the Projection Theorem is equivalently stated in terms of *orthogonal* direct sum up to unitary equivalence, $\mathcal{H} \cong \mathcal{M} \oplus \mathcal{M}^\perp$, often written as

$$\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp.$$

This leads to the notation $\mathcal{M}^\perp = \mathcal{H} \ominus \mathcal{M}$, if $\mathcal{M}$ is a *subspace* of $\mathcal{H}$.

For a nonempty subset A of a normed space $\mathcal{X}$, let $\operatorname{span} A$ denote the *span* (or *linear span*) of A: the linear manifold of $\mathcal{X}$ consisting of all (finite) linear combinations of vectors of A; equivalently, the smallest linear manifold of $\mathcal{X}$ that includes A (i.e., the intersection of all linear manifolds of $\mathcal{X}$ that include A). Its closure $(\operatorname{span} A)^-$, denoted by $\bigvee A$ (and sometimes also called the *span* of A), is a subspace of $\mathcal{X}$. Let $\{\mathcal{M}_\gamma\}_{\gamma \in \Gamma}$ be a family of subspaces of $\mathcal{X}$ indexed by an arbitrary (not necessarily countable) nonempty index set Γ. The subspace $\left[\operatorname{span} \left(\bigcup_{\gamma \in \Gamma} \mathcal{M}_\gamma \right) \right]^-$ of $\mathcal{X}$ is the *topological sum* of $\{\mathcal{M}_\gamma\}_{\gamma \in \Gamma}$, usually denoted by $\left(\sum_{\gamma \in \Gamma} \mathcal{M}_\gamma \right)^-$ or $\bigvee_{\gamma \in \Gamma} \mathcal{M}_\gamma$. That is,

$$\left(\sum_{\gamma \in \Gamma} \mathcal{M}_\gamma \right)^- = \left[\operatorname{span} \left(\bigcup_{\gamma \in \Gamma} \mathcal{M}_\gamma \right) \right]^- = \bigvee \left(\bigcup_{\gamma \in \Gamma} \mathcal{M}_\gamma \right) = \bigvee_{\gamma \in \Gamma} \mathcal{M}_\gamma.$$

A pivotal result of Hilbert space theory is the ORTHOGONAL STRUCTURE THEOREM, which reads as follows. *Let $\{\mathcal{M}_\gamma\}_{\gamma \in \Gamma}$ be a family of pairwise orthogonal subspaces of a Hilbert space $\mathcal{H}$. For every $x \in \left(\sum_{\gamma \in \Gamma} \mathcal{M}_\gamma \right)^-$ there is a unique summable family $\{u_\gamma\}_{\gamma \in \Gamma}$ of vectors $u_\gamma \in \mathcal{M}_\gamma$ such that*

$$x = \sum_{\gamma \in \Gamma} u_\gamma.$$

Moreover, $\|x\|^2 = \sum_{\gamma \in \Gamma} \|u_\gamma\|^2$. Conversely, if $\{u_\gamma\}_{\gamma \in \Gamma}$ is a square-summable family of vectors in $\mathcal{H}$ with $u_\gamma \in \mathcal{M}_\gamma$ for each $\gamma \in \Gamma$, then $\{u_\gamma\}_{\gamma \in \Gamma}$ is summable and $\sum_{\gamma \in \Gamma} u_\gamma$ lies in $\left(\sum_{\gamma \in \Gamma} \mathcal{M}_\gamma \right)^-$. (See, e.g., [66, Section 5.5].)

For the definition of *summable family* and *square-summable family* see [66, Definition 5.26]. A summable family of vectors in a normed space has only a countable number of nonzero vectors, and so the sum $\sum_{\gamma \in \Gamma} u_\gamma$ has only a countable number of nonzero summands u_γ (see, e.g., [66, Corollary 5.28]).

Take the direct sum $\bigoplus_{\gamma \in \Gamma} \mathcal{M}_\gamma$ of pairwise orthogonal subspaces $\mathcal{M}_\gamma$ of a Hilbert space $\mathcal{H}$ made up of square-summable families of vectors. That is, $\mathcal{M}_\alpha \perp \mathcal{M}_\beta$ for $\alpha \neq \beta$, and $x = \{x_\gamma\}_{\gamma \in \Gamma} \in \bigoplus_{\gamma \in \Gamma} \mathcal{M}_\gamma$ if and only if each $x_\gamma \in \mathcal{M}_\gamma$ and $\sum_{\gamma \in \Gamma} \|x_\gamma\|^2 < \infty$. This is a Hilbert space with inner product

$$\langle x \,;\, y \rangle = \sum\nolimits_{\gamma \in \Gamma} \langle x_\gamma \,;\, y_\gamma \rangle$$

for every $x = \{x_\gamma\}_{\gamma \in \Gamma}$ and $y = \{y_\gamma\}_{\gamma \in \Gamma}$ in $\bigoplus_{\gamma \in \Gamma} \mathcal{M}_\gamma$. A consequence (a restatement, actually) of the Orthogonal Structure Theorem says that the orthogonal direct sum $\bigoplus_{\gamma \in \Gamma} \mathcal{M}_\gamma$ and the topological sum $\left(\sum_{\gamma \in \Gamma} \mathcal{M}_\gamma \right)^-$ are unitarily equivalent Hilbert spaces. That is,

$$\bigoplus\nolimits_{\gamma \in \Gamma} \mathcal{M}_\gamma \cong \left(\sum\nolimits_{\gamma \in \Gamma} \mathcal{M}_\gamma \right)^-.$$

If, in addition, $\{\mathcal{M}_\gamma\}_{\gamma \in \Gamma}$ spans $\mathcal{H}$ (i.e., $\left(\sum_{\gamma \in \Gamma} \mathcal{M}_\gamma \right)^- = \bigvee_{\gamma \in \Gamma} \mathcal{M}_\gamma = \mathcal{H}$), then it is usual to express the above identification by writing

$$\mathcal{H} = \bigoplus\nolimits_{\gamma \in \Gamma} \mathcal{M}_\gamma.$$

Similarly, the orthogonal direct sum $\bigoplus_{\gamma \in \Gamma} \mathcal{H}_\gamma$ of a collection of Hilbert spaces $\{\mathcal{H}_\gamma\}_{\gamma \in \Gamma}$ (not necessarily subspaces of any larger Hilbert space) consists of square-summable families $\{x_\gamma\}_{\gamma \in \Gamma}$ (i.e., $\sum_{\gamma \in \Gamma} \|x_\gamma\|^2 < \infty$), with each x_γ in $\mathcal{H}_\gamma$. This is a Hilbert space with inner product given as above.

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. The *direct sum* of $T \in \mathcal{B}[\mathcal{H}]$ and $S \in \mathcal{B}[\mathcal{K}]$ is the operator $T \oplus S \in \mathcal{B}[\mathcal{H} \oplus \mathcal{K}]$ such that $(T \oplus S)(x, y) = (Tx, Sy)$ for every $(x, y) \in \mathcal{H} \oplus \mathcal{K}$. If $\{T_\gamma\}_{\gamma \in \Gamma}$ is a family of operators with $T_\gamma \in \mathcal{B}[\mathcal{H}_\gamma]$, then their *direct sum* $T \colon \bigoplus_{\gamma \in \Gamma} \mathcal{H}_\gamma \to \bigoplus_{\gamma \in \Gamma} \mathcal{H}_\gamma$ is the mapping such that

$$T\{x_\gamma\}_{\gamma \in \Gamma} = \{T_\gamma x_\gamma\}_{\gamma \in \Gamma} \quad \text{for every} \quad \{x_\gamma\}_{\gamma \in \Gamma} \in \bigoplus\nolimits_{\gamma \in \Gamma} \mathcal{H}_\gamma,$$

which is an operator in $\mathcal{B}\left[\bigoplus_{\gamma \in \Gamma} \mathcal{H}_\gamma \right]$ with $\|T\| = \sup_{\gamma \in \Gamma} \|T_\gamma\|$, denoted by

$$T = \bigoplus\nolimits_{\gamma \in \Gamma} T_\gamma.$$

1.4 Orthogonal Projections

A function $F \colon X \to X$ of a set X into itself is *idempotent* if $F^2 = F$, where F^2 stands for the composition of F with itself. A *projection* E is an idempotent (i.e., $E = E^2$) linear transformation $E \colon \mathcal{X} \to \mathcal{X}$ of a linear space $\mathcal{X}$ into itself. If E is a projection, then so is $I - E$, which is referred to as the *complementary projection* of E, and their null spaces and ranges are related as follows:

$$\mathcal{R}(E) = \mathcal{N}(I - E) \quad \text{and} \quad \mathcal{N}(E) = \mathcal{R}(I - E).$$

The range of a projection is the set of all its fixed points,

$$\mathcal{R}(E) = \big\{x \in \mathcal{X}: \ Ex = x\big\},$$

which forms with the kernel a pair of algebraic complements,

$$\mathcal{R}(E) + \mathcal{N}(E) = \mathcal{X} \quad \text{and} \quad \mathcal{R}(E) \cap \mathcal{N}(E) = \{0\}.$$

An *orthogonal projection* $E: \mathcal{X} \to \mathcal{X}$ on an inner product space $\mathcal{X}$ is a projection such that $\mathcal{R}(E) \perp \mathcal{N}(E)$. If E is an orthogonal projection on $\mathcal{X}$, then so is the complementary projection $I - E: \mathcal{X} \to \mathcal{X}$. *Every orthogonal projection is bounded* (i.e., *continuous* — $E \in \mathcal{B}[\mathcal{H}]$). In fact, every orthogonal projection is a *contraction*. Indeed, every $x \in \mathcal{X}$ can be written as $x = u + v$ with $u \in \mathcal{R}(E)$ and $v \in \mathcal{N}(E)$, so that $\|Ex\|^2 = \|Eu + Ev\|^2 = \|Eu\|^2 = \|u\|^2 \leq \|x\|^2$ since $\|x\|^2 = \|u\|^2 + \|v\|^2$ by the Pythagorean Theorem (once $u \perp v$). Hence $\mathcal{R}(E)$ is a subspace (i.e., it is closed, because $\mathcal{R}(E) = \mathcal{N}(I - E)$ and $I - E$ is bounded). Moreover, $\mathcal{N}(E) = \mathcal{R}(E)^{\perp}$; equivalently, $\mathcal{R}(E) = \mathcal{N}(E)^{\perp}$. Conversely, if any of these properties holds for a projection E, then it is an orthogonal projection.

Two orthogonal projections E_1 and E_2 on $\mathcal{X}$ are said to be *orthogonal to each other* (or *mutually orthogonal*) if $\mathcal{R}(E_1) \perp \mathcal{R}(E_2)$ (which is equivalent to saying that $E_1 E_2 = E_2 E_1 = O$). Let Γ be an arbitrary index set (not necessarily countable). If $\{E_\gamma\}_{\gamma \in \Gamma}$ is a family of orthogonal projections on an inner product space $\mathcal{X}$ that are orthogonal to each other ($\mathcal{R}(E_\alpha) \perp \mathcal{R}(E_\beta)$ for $\alpha \neq \beta$), then $\{E_\gamma\}_{\gamma \in \Gamma}$ is an *orthogonal family of orthogonal projections* on $\mathcal{X}$. An *orthogonal sequence of orthogonal projections* $\{E_k\}_{k=0}^{\infty}$ is similarly defined. If $\{E_\gamma\}_{\gamma \in \Gamma}$ is an orthogonal family of orthogonal projections and

$$\sum\nolimits_{\gamma \in \Gamma} E_\gamma x = x \ \text{ for every } \ x \in \mathcal{X},$$

then $\{E_\gamma\}_{\gamma \in \Gamma}$ is called a *resolution of the identity* on $\mathcal{X}$. (Recall that for each $x \in \mathcal{X}$ the sum $x = \sum_{\gamma \in \Gamma} E_\gamma x$ has only a countable number of nonzero vectors.) If $\{E_k\}_{k=0}^{\infty}$ is an infinite sequence, then the above identity in $\mathcal{X}$ means convergence in the strong operator topology:

$$\sum_{k=0}^{n} E_k \xrightarrow{\ s\ } I \ \text{ as } \ n \to \infty.$$

If $\{E_k\}_{k=0}^{n}$ is a finite family, then the above identity in $\mathcal{X}$ obviously coincides with the identity $\sum_{k=0}^{n} E_k = I$ in $\mathcal{B}[\mathcal{X}]$ (e.g., $\{E_1, E_2\}$ is a resolution of the identity on $\mathcal{X}$ whenever $E_2 = I - E_1$ is the complementary projection of the orthogonal projection E_1 on $\mathcal{X}$).

The Projection Theorem and the Orthogonal Structure Theorem of the previous section can be written in terms of orthogonal projections. Indeed, *for every subspace $\mathcal{M}$ of a Hilbert space $\mathcal{H}$ there exists a unique orthogonal*

projection $E \in \mathcal{B}[\mathcal{H}]$ such that $\mathcal{R}(E) = \mathcal{M}$, which is referred to as *the orthogonal projection on* $\mathcal{M}$. This is an equivalent version of the Projection Theorem. An equivalent version of the Orthogonal Structure Theorem reads as follows. If $\{E_\gamma\}_{\gamma \in \Gamma}$ is a resolution of the identity on $\mathcal{H}$, then

$$\mathcal{H} = \left(\sum\nolimits_{\gamma \in \Gamma} \mathcal{R}(E_\gamma)\right)^-.$$

Conversely, if $\{\mathcal{M}_\gamma\}_{\gamma \in \Gamma}$ *is a family of pairwise orthogonal subspaces of* $\mathcal{H}$ *such that* $\mathcal{H} = \left(\sum_{\gamma \in \Gamma} \mathcal{M}_\gamma\right)^-$, *then the family of orthogonal projections* $\{E_\gamma\}_{\gamma \in \Gamma}$, *with each* $E_\gamma \in \mathcal{B}[\mathcal{H}]$ *on each* $\mathcal{M}_\gamma$ (i.e., *with* $\mathcal{R}(E_\gamma) = \mathcal{M}_\gamma$), *is a resolution of the identity on* $\mathcal{H}$. Again, since $\{\mathcal{R}(E_\gamma)\}_{\gamma \in \Gamma}$ is a family of pairwise orthogonal subspaces, the above orthogonal decomposition of $\mathcal{H}$ is unitarily equivalent to the orthogonal direct sum $\bigoplus_{\gamma \in \Gamma} \mathcal{R}(E_\gamma)$, that is, $\left(\sum_{\gamma \in \Gamma} \mathcal{R}(E_\gamma)\right)^- \cong \bigoplus_{\gamma \in \Gamma} \mathcal{R}(E_\gamma)$, which is commonly written as

$$\mathcal{H} = \bigoplus\nolimits_{\gamma \in \Gamma} \mathcal{R}(E_\gamma).$$

(See, e.g., [66, Theorems 5.52 and 5.59].) Let $\{E_\gamma\}_{\gamma \in \Gamma}$ be a resolution of the identity on a nonzero Hilbert space made up of nonzero projections, let $\{\lambda_\gamma\}_{\gamma \in \Gamma}$ be a similarly indexed family of scalars, and set

$$\mathcal{D}(T) = \big\{x \in \mathcal{H} \colon \{\lambda_\gamma E_\gamma x\}_{\gamma \in \Gamma} \text{ is a summable family of vectors in } \mathcal{H}\big\}.$$

The mapping $T \colon \mathcal{D}(T) \to \mathcal{H}$, defined by

$$Tx = \sum\nolimits_{\gamma \in \Gamma} \lambda_\gamma E_\gamma x \quad \text{for every } x \in \mathcal{D}(T),$$

is called a *weighted sum of projections*. It can be shown that the domain $\mathcal{D}(T)$ of a weighted sum of projections is a linear manifold of $\mathcal{H}$ and that T is a linear transformation. Moreover, *T is bounded if and only if* $\{\lambda_\gamma\}_{\gamma \in \Gamma}$ *is a bounded family of scalars, which happens if and only if* $\mathcal{D}(T) = \mathcal{H}$. In this case $T \in \mathcal{B}[\mathcal{H}]$ and $\|T\| = \sup_{\gamma \in \Gamma} |\lambda_\gamma|$. (See, e.g., [66, Proposition 5.61].)

1.5 Adjoint Operator

Throughout the text, $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces. Take any $T \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$. The *adjoint* T^* of T is the unique mapping of $\mathcal{K}$ into $\mathcal{H}$ such that

$$\langle Tx \,; y \rangle = \langle x \,; T^*y \rangle \quad \text{for every } x \in \mathcal{H} \text{ and } y \in \mathcal{K}.$$

In fact, $T^* \in \mathcal{B}[\mathcal{K}, \mathcal{H}]$ (i.e., T^* is bounded and linear), $T^{**} = T$, and

$$\|T^*\|^2 = \|T^*T\| = \|TT^*\| = \|T\|^2.$$

Moreover, if $\mathcal{Z}$ is a Hilbert space and $S \in \mathcal{B}[\mathcal{K}, \mathcal{Z}]$, then $ST \in \mathcal{B}[\mathcal{H}, \mathcal{Z}]$ is such that $(ST)^* = T^*S^*$ (see, e.g., [66, Proposition 5.65 and Section 5.12]).

Lemma 1.4. *For every $T \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$,*

$$\mathcal{N}(T) = \mathcal{R}(T^*)^\perp = \mathcal{N}(T^*T) \quad \text{and} \quad \mathcal{R}(T)^- = \mathcal{N}(T^*)^\perp = \mathcal{R}(TT^*)^-.$$

Proof. $x \in \mathcal{R}(T^*)^\perp$ if and only if $\langle x \, ; T^*y \rangle = 0$ or, equivalently, $\langle Tx \, ; y \rangle = 0$, for every y in $\mathcal{K}$. This means that $Tx = 0$; that is, $x \in \mathcal{N}(T)$. Hence

$$\mathcal{R}(T^*)^\perp = \mathcal{N}(T).$$

Since $\|Tx\|^2 = \langle Tx \, ; Tx \rangle = \langle T^*Tx \, ; x \rangle$ for every x in $\mathcal{H}$, it follows that $\mathcal{N}(T^*T) \subseteq \mathcal{N}(T)$. But $\mathcal{N}(T) \subseteq \mathcal{N}(T^*T)$ trivially, and so

$$\mathcal{N}(T) = \mathcal{N}(T^*T).$$

This completes the proof of the first identities. Since these hold for every T in $\mathcal{B}[\mathcal{H}, \mathcal{K}]$, they also hold for T^* in $\mathcal{B}[\mathcal{K}, \mathcal{H}]$ and for TT^* in $\mathcal{B}[\mathcal{K}]$. Thus

$$\begin{aligned} \mathcal{R}(T)^- &= \mathcal{R}(T^{**})^{\perp\perp} = \mathcal{N}(T^*)^\perp = \mathcal{N}(T^{**}T^*)^\perp \\ &= \mathcal{N}(TT^*)^\perp = \mathcal{R}((TT^*)^*)^{\perp\perp} = \mathcal{R}(T^{**}T^*)^{\perp\perp} = \mathcal{R}(TT^*)^-, \end{aligned}$$

because $\mathcal{M}^{\perp\perp} = \mathcal{M}^-$ for every linear manifold $\mathcal{M}$ and $T^{**} = T$. $\square$

Lemma 1.5. $\mathcal{R}(T)$ *is closed in* $\mathcal{K}$ *if and only if* $\mathcal{R}(T^*)$ *is closed in* $\mathcal{H}$.

Proof. Set $T_\perp = T|_{\mathcal{N}(T)^\perp} \in \mathcal{B}[\mathcal{N}(T)^\perp, \mathcal{K}]$, the restriction of T to $\mathcal{N}(T)^\perp$. Recall that $\mathcal{H} = \mathcal{N}(T) + \mathcal{N}(T)^\perp$. Thus every $x \in \mathcal{H}$ can be written as $x = u + v$ with $u \in \mathcal{N}(T)$ and $v \in \mathcal{N}(T)^\perp$. If $y \in \mathcal{R}(T)$, then $y = Tx = Tu + Tv = Tv = T|_{\mathcal{N}(T)^\perp} v$ for some $x \in \mathcal{H}$, and so $y \in \mathcal{R}(T|_{\mathcal{N}(T)^\perp})$. Hence $\mathcal{R}(T) \subseteq \mathcal{R}(T|_{\mathcal{N}(T)^\perp})$. Since $\mathcal{R}(T|_{\mathcal{N}(T)^\perp}) \subseteq \mathcal{R}(T)$, it follows that

$$\mathcal{R}(T_\perp) = \mathcal{R}(T) \quad \text{and} \quad \mathcal{N}(T_\perp) = \{0\}.$$

If $\mathcal{R}(T) = \mathcal{R}(T)^-$, then take the inverse $T_\perp^{-1}$ in $\mathcal{B}[\mathcal{R}(T), \mathcal{N}(T)^\perp]$ (Theorem 1.2). For any $w \in \mathcal{N}(T)^\perp$ consider the functional $f_w : \mathcal{R}(T) \to \mathbb{C}$ defined by

$$f_w(y) = \langle T_\perp^{-1} y \, ; w \rangle$$

for every $y \in \mathcal{R}(T)$. Since $T_\perp^{-1}$ is linear and the inner product is linear in the first argument, it follows that f is linear. Since $|f_w(y)| \leq \|T_\perp^{-1}\| \|w\| \|y\|$ for every $y \in \mathcal{R}(T)$, it follows that f is bounded. Moreover, $\mathcal{R}(T)$ is a subspace of the Hilbert space $\mathcal{K}$ (it is closed in $\mathcal{K}$), and so a Hilbert space itself. Then the Riesz Representation Theorem in Hilbert spaces (see, e.g., [66, Theorem 5.62]) says that there exists z_w in the Hilbert space $\mathcal{R}(T)$ such that

$$f_w(y) = \langle y \, ; z_w \rangle$$

for every $y \in \mathcal{R}(T)$. Thus, for an arbitrary $x \in \mathcal{H}$, so that $x = u + v$ with $u \in \mathcal{N}(T)$ and $v \in \mathcal{N}(T)^{\perp}$, we get

$$\begin{aligned} \langle x\,;T^*z_w \rangle = \langle Tx\,;z_w \rangle &= \langle Tu\,;z_w \rangle + \langle Tv\,;z_w \rangle = \langle Tv\,;z_w \rangle \\ &= f_w(Tv) = \langle T_{\perp}^{-1}Tv\,;w \rangle = \langle T_{\perp}^{-1}T_{\perp}v\,;w \rangle \\ &= \langle v\,;w \rangle = \langle u\,;w \rangle + \langle v\,;w \rangle = \langle x\,;w \rangle. \end{aligned}$$

Hence $\langle x\,;T^*z_w - w \rangle = 0$ for every $x \in \mathcal{H}$, which means that $T^*z_w = w$. Thus $w \in \mathcal{R}(T^*)$. This shows that $\mathcal{N}(T)^{\perp} \subseteq \mathcal{R}(T^*)$. On the other hand, $\mathcal{R}(T^*) \subseteq \mathcal{R}(T^*)^- = \mathcal{N}(T)^{\perp}$ by Lemma 1.4 (since $T^{**} = T$) so that

$$\mathcal{R}(T^*) = \mathcal{R}(T^*)^-.$$

Therefore $\mathcal{R}(T) = \mathcal{R}(T)^-$ implies $\mathcal{R}(T^*) = \mathcal{R}(T^*)^-$, and so the converse also holds because $T^{**} = T$. That is,

$$\mathcal{R}(T) = \mathcal{R}(T)^- \quad \text{if and only if} \quad \mathcal{R}(T^*) = \mathcal{R}(T^*)^-. \qquad \square$$

Let $T \in \mathcal{B}[\mathcal{H}]$ be an operator on a Hilbert space $\mathcal{H}$ and let $\mathcal{M}$ be a subspace of $\mathcal{H}$. If $\mathcal{M}$ and its orthogonal complement $\mathcal{M}^{\perp}$ are both T-invariant (i.e., if $T(\mathcal{M}) \subseteq \mathcal{M}$ and $T(\mathcal{M}^{\perp}) \subseteq \mathcal{M}^{\perp}$), then we say that $\mathcal{M}$ *reduces* T (or $\mathcal{M}$ is a *reducing subspace* for T). An operator is *reducible* if it has a nontrivial reducing subspace (otherwise it is called *irreducible*), and *reductive* if all its invariant subspaces are reducing. Since $\mathcal{M} \neq \{0\}$ if and only if $\mathcal{M}^{\perp} \neq \mathcal{H}$, and $\mathcal{M}^{\perp} \neq \{0\}$ if and only if $\mathcal{M} \neq \mathcal{H}$, it follows that an operator T on a Hilbert space $\mathcal{H}$ is reducible if there exists a subspace $\mathcal{M}$ of $\mathcal{H}$ such that both $\mathcal{M}$ and $\mathcal{M}^{\perp}$ are nonzero and T-invariant or, equivalently, if $\mathcal{M}$ is nontrivial and invariant for both T and T^*, as we shall see next.

Lemma 1.6. *Take an operator T on a Hilbert space $\mathcal{H}$. A subspace $\mathcal{M}$ of $\mathcal{H}$ is T-invariant if and only if $\mathcal{M}^{\perp}$ is T^*-invariant. $\mathcal{M}$ reduces T if and only if $\mathcal{M}$ is invariant for both T and T^*, which implies that $(T|_{\mathcal{M}})^* = T^*|_{\mathcal{M}}$.*

Proof. Take any vector $y \in \mathcal{M}^{\perp}$. If $Tx \in \mathcal{M}$ whenever $x \in \mathcal{M}$, then $\langle x\,;T^*y \rangle = \langle Tx\,;y \rangle = 0$ for every $x \in \mathcal{M}$, and so $T^*y \perp \mathcal{M}$; that is, $T^*y \in \mathcal{M}^{\perp}$. Thus $T(\mathcal{M}) \subseteq \mathcal{M}$ implies $T^*(\mathcal{M}^{\perp}) \subseteq \mathcal{M}^{\perp}$. Since this holds for every operator in $\mathcal{B}[\mathcal{H}]$, $T^*(\mathcal{M}^{\perp}) \subseteq \mathcal{M}^{\perp}$ implies $T^{**}(\mathcal{M}^{\perp\perp}) \subseteq \mathcal{M}^{\perp\perp}$, and hence $T(\mathcal{M}) \subseteq \mathcal{M}$ (because $T^{**} = T$ and $\mathcal{M}^{\perp\perp} = \mathcal{M}^- = \mathcal{M}$). Then $T(\mathcal{M}) \subseteq \mathcal{M}$ if and only if $T^*(\mathcal{M}^{\perp}) \subseteq \mathcal{M}^{\perp}$, and so $T(\mathcal{M}^{\perp}) \subseteq \mathcal{M}^{\perp}$ if and only if $T^*(\mathcal{M}) \subseteq \mathcal{M}$. Therefore $T(\mathcal{M}) \subseteq \mathcal{M}$ and $T(\mathcal{M}^{\perp}) \subseteq \mathcal{M}^{\perp}$ if and only if $T(\mathcal{M}) \subseteq \mathcal{M}$ and $T^*(\mathcal{M}) \subseteq \mathcal{M}$, and so $\langle (T|_{\mathcal{M}})x\,;y \rangle = \langle Tx\,;y \rangle = \langle x\,;T^*y \rangle = \langle x\,;(T^*|_{\mathcal{M}})y \rangle$ for $x, y \in \mathcal{M}$. $\square$

An operator $T \in \mathcal{B}[\mathcal{H}]$ on a Hilbert space $\mathcal{H}$ is *self-adjoint* (or *Hermitian*) if it coincides with its adjoint (i.e., if $T^* = T$). A characterization of self-adjoint operators reads as follows: *on a complex Hilbert space $\mathcal{H}$, an operator T is self-adjoint if and only if $\langle Tx\,;x \rangle \in \mathbb{R}$ for every vector $x \in \mathcal{H}$. Moreover, if*

$T \in \mathcal{B}[\mathcal{H}]$ is self-adjoint, then $T = O$ if and only if $\langle Tx \, ; x \rangle = 0$ *for all* $x \in \mathcal{H}$. (See, e.g., [66, Proposition 5.79 and Corollary 5.80].) A self-adjoint operator T is *nonnegative* (notation: $O \leq T$) if $0 \leq \langle Tx \, ; x \rangle$ for every $x \in \mathcal{H}$, and it is *positive* (notation: $O < T$) if $0 < \langle Tx \, ; x \rangle$ for every nonzero $x \in \mathcal{H}$. An invertible positive operator T in $\mathcal{G}[\mathcal{H}]$ is called *strictly positive* (notation: $O \prec T$). For every T in $\mathcal{B}[\mathcal{H}]$, the operators T^*T and TT^* in $\mathcal{B}[\mathcal{H}]$ are always nonnegative (reason: $0 \leq \|Tx\|^2 = \langle Tx \, ; Tx \rangle = \langle T^*Tx \, ; x \rangle$ for every $x \in \mathcal{H}$). If S and T are operators in $\mathcal{B}[\mathcal{H}]$, then we write $S \leq T$ if $T - S$ is nonnegative. Similarly, we also write $S < T$ and $S \prec T$ if $T - S$ is positive or strictly positive. Thus, *if* $T - S$ *is self-adjoint* (in particular, if both are), *then* $S = T$ *if and only if* $S \leq T$ *and* $T \leq S$.

1.6 Normal Operators

Let S and T be operators on the same space $\mathcal{X}$. We say that they *commute* if $ST = TS$. An operator $T \in \mathcal{B}[\mathcal{H}]$ on a Hilbert space $\mathcal{H}$ is *normal* if it commutes with its adjoint (i.e., if $TT^* = T^*T$, or $O = T^*T - TT^*$). An operator $T \in \mathcal{B}[\mathcal{H}]$ is *hyponormal* if $TT^* \leq T^*T$ (i.e., if $O \leq T^*T - TT^*$), which is equivalent to saying that $(\lambda I - T)(\lambda I - T)^* \leq (\lambda I - T)^*(\lambda I - T)$ for every scalar $\lambda \in \mathbb{C}$. An operator $T \in \mathcal{B}[\mathcal{H}]$ is *cohyponormal* if its adjoint is hyponormal (i.e., if $T^*T \leq TT^*$, or $O \leq TT^* - T^*T$). Since $T^*T - TT^*$ is always self-adjoint, it follows that T is normal if and only if it is both hyponormal and cohyponormal. Thus *every normal operator is hyponormal*. If it is either hyponormal or cohyponormal, then it is called *seminormal*.

Theorem 1.7. $T \in \mathcal{B}[\mathcal{H}]$ *is hyponormal if and only if* $\|T^*x\| \leq \|Tx\|$ *for every* $x \in \mathcal{H}$. *Moreover, the following assertions are pairwise equivalent.*

(a) *T is normal.*

(b) $\|T^*x\| = \|Tx\|$ *for every* $x \in \mathcal{H}$.

(c) *T^n is normal for every positive integer n.*

(d) $\|T^{*n}x\| = \|T^nx\|$ *for every* $x \in \mathcal{H}$ *and every* $n \geq 1$.

Proof. Take any $T \in \mathcal{B}[\mathcal{H}]$. The characterization for hyponormal operators goes as follows: $TT^* \leq T^*T$ if and only if $\langle TT^*x \, ; x \rangle \leq \langle T^*Tx \, ; x \rangle$ or, equivalently, $\|T^*x\|^2 \leq \|Tx\|^2$, for every $x \in \mathcal{H}$. Since T is normal if and only if it is both hyponormal and cohyponormal, it follows that (a) and (b) are equivalent. Thus, as $T^{*n} = T^{n*}$ for each $n \geq 1$, (c) and (d) are equivalent. If T^* commutes with T, then it commutes with T^n and, dually, T^n commutes with $T^{*n} = T^{n*}$. Hence (a) implies (c), and (d) implies (b) trivially. $\square$

In spite of the equivalence between (a) and (c) above, the square of a hyponormal is not necessarily hyponormal. It is clear that every self-adjoint

operator is normal, and so is every nonnegative operator. Moreover, normality distinguishes the orthogonal projections among the projections.

Theorem 1.8. *If $E \in \mathcal{B}[\mathcal{H}]$ is a nonzero projection, then the following assertions are pairwise equivalent.*

(a) *E is an orthogonal projection.*

(b) *E is nonnegative.*

(c) *E is self-adjoint.*

(d) *E is normal.*

(e) $\|E\| = 1$.

(f) $\|E\| \leq 1$.

Proof. Let E be a projection (i.e., a linear idempotent). If E is an orthogonal projection (i.e., $\mathcal{R}(E) \perp \mathcal{N}(E)$), then $I - E$ is again an orthogonal projection, thus bounded, and so $\mathcal{R}(E) = \mathcal{N}(I - E)$ is closed. Hence $\mathcal{R}(E^*)$ is closed by Lemma 1.5. Moreover, $\mathcal{R}(E) = \mathcal{N}(E)^\perp$. But $\mathcal{N}(E)^\perp = \mathcal{R}(E^*)^-$ by Lemma 1.4 (since $E^{**} = E$). Then $\mathcal{R}(E) = \mathcal{R}(E^*)$, and so for every $x \in \mathcal{H}$ there is a $z \in \mathcal{H}$ such that $Ex = E^* z$. Therefore, $E^* E x = (E^*)^2 z = (E^2)^* z = E^* z = Ex$. That is, $E^* E = E$. This implies that E is nonnegative (it is self-adjoint, and $E^* E$ is always nonnegative). Outcome:

$$\mathcal{R}(E) \perp \mathcal{N}(E) \quad \Longrightarrow \quad E \geq O \quad \Longrightarrow \quad E^* = E \quad \Longrightarrow \quad E^* E = EE^*.$$

Conversely, if E is normal, then $\|E^* x\| = \|Ex\|$ for every x in $\mathcal{H}$ (Theorem 1.7) so that $\mathcal{N}(E^*) = \mathcal{N}(E)$. Hence $\mathcal{R}(E) = \mathcal{N}(E^*)^\perp = \mathcal{N}(E)^\perp$ by Lemma 1.4 (because $\mathcal{R}(E)$ is closed). Therefore, $\mathcal{R}(E) \perp \mathcal{N}(E)$. That is,

$$E^* E = EE^* \quad \Longrightarrow \quad \mathcal{R}(E) \perp \mathcal{N}(E).$$

If $O \neq E = E^*$, then $\|E\|^2 = \|E^* E\| = \|E^2\| = \|E\| \neq 0$, and so $\|E\| = 1$:

$$E = E^* \quad \Longrightarrow \quad \|E\| = 1 \quad \Longrightarrow \quad \|E\| \leq 1.$$

Take any $v \in \mathcal{N}(E)^\perp$ so that $(I - E)v \in \mathcal{N}(E)$ (since $\mathcal{R}(I - E) = \mathcal{N}(E)$). Hence $(I - E)v \perp v$ so that $0 = \langle (I - E)v \,;v \rangle = \|v\|^2 - \langle Ev \,;v \rangle$. If $\|E\| \leq 1$, then $\|v\|^2 = \langle Ev \,;v \rangle \leq \|Ev\|\|v\| \leq \|E\|\|v\|^2 \leq \|v\|^2$. Thus $\|Ev\| = \|v\| = \langle Ev \,;v \rangle^{\frac{1}{2}}$. So $\|(I - E)v\|^2 = \|Ev - v\|^2 = \|Ev\|^2 - 2\operatorname{Re}\langle Ev \,;v \rangle + \|v\|^2 = 0$, and hence $v \in \mathcal{N}(I - E) = \mathcal{R}(E)$. Then $\mathcal{N}(E)^\perp \subseteq \mathcal{R}(E)$, which implies that $\mathcal{R}(E^*) \subseteq \mathcal{N}(E^*)^\perp$ by Lemma 1.4, and hence $\mathcal{R}(E^*) \perp \mathcal{N}(E^*)$. But E^* is a projection whenever E is ($E = E^2$ implies $E^* = E^{2*} = E^{*2}$). Thus E^* is an orthogonal projection so that E^* is self-adjoint, and so is E. Therefore,

$$\|E\| \leq 1 \quad \Longrightarrow \quad E = E^*. \qquad \square$$

Corollary 1.9. *Every bounded weighted sum of projections is normal.*

Proof. Let $T \in \mathcal{B}[\mathcal{H}]$ be a weighted sum of projections, which means that $Tx = \sum_{\gamma \in \Gamma} \lambda_\gamma E_\gamma x$ for every x in the Hilbert space $\mathcal{H}$, where $\{\lambda_\gamma\}_{\gamma \in \Gamma}$ is a bounded family of scalars and $\{E_\gamma\}_{\gamma \in \Gamma}$ is a resolution of the identity on $\mathcal{H}$. Since each $E_\gamma^* = E_\gamma$ and since $\mathcal{R}(E_\alpha) \perp \mathcal{R}(E_\beta)$ for $\alpha \neq \beta$, it is readily verified that the adjoint T^* of T is given by $T^*x = \sum_{\gamma \in \Gamma} \overline{\lambda}_\gamma E_\gamma x$ for every x in $\mathcal{H}$ (i.e., $\langle Tx \, ; y \rangle = \langle x \, ; T^*y \rangle$ for every x, y in $\mathcal{H}$). Moreover, since $E_\gamma^2 = E_\gamma$ and since $E_\alpha E_\beta = E_\beta E_\alpha = O$ for $\alpha \neq \beta$, it is also readily verified that $TT^*x = T^*Tx$ for every $x \in \mathcal{H}$. Thus T is normal. $\square$

An *isometry* between metric spaces is a map that preserves distance, and so every isometry is injective. A linear transformation T between normed spaces is an isometry if and only if it preserves norm ($\|Tx\| = \|x\|$ for every x) and, between inner product spaces, if and only if it preserves inner product ($\langle Tx \, ; Ty \rangle = \langle x \, ; y \rangle$ for every x, y in $\mathcal{H}$). A transformation $T \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$ between Hilbert spaces is an *isometry* if and only if $T^*T = I$ (identity on $\mathcal{H}$), and a *coisometry* if its adjoint is an isometry (i.e., if $TT^* = I$; identity on $\mathcal{K}$). It is a *unitary transformation* if it is an isometry and a coisometry; that is, if it is an invertible transformation with $T^{-1} = T^*$ or, equivalently, an invertible isometry, which means a surjective isometry. Thus a *unitary operator* $T \in \mathcal{B}[\mathcal{H}]$ is precisely a normal isometry: $TT^* = T^*T = I$; and so normality distinguishes the unitary operators among the isometries.

Lemma 1.10. *If $\mathcal{X}$ is a normed space and $T \in \mathcal{B}[\mathcal{X}]$, then the real-valued sequence $\{\|T^n\|^{\frac{1}{n}}\}$ converges in $\mathbb{R}$.*

Proof. Take any positive integer m. Every positive integer n can be written as $n = m\,p_n + q_n$ for some nonnegative integers p_n, q_n with $q_n < m$. Hence

$$\|T^n\| = \|T^{mp_n + q_n}\| = \|T^{mp_n} T^{q_n}\| \leq \|T^{mp_n}\| \|T^{q_n}\| \leq \|T^m\|^{p_n} \|T^{q_n}\|.$$

Now suppose $T \neq 0$ to avoid trivialities, set $\mu = \max_{0 \leq k \leq m-1}\{\|T^k\|\} \neq 0$, and recall that $q_n \leq m - 1$. Then

$$\|T^n\|^{\frac{1}{n}} \leq \|T^m\|^{\frac{p_n}{n}} \mu^{\frac{1}{n}} = \mu^{\frac{1}{n}} \|T^m\|^{\frac{1}{m} - \frac{q_n}{mn}}.$$

Since $\mu^{\frac{1}{n}} \to 1$ and $\|T^m\|^{\frac{1}{m} - \frac{q_n}{mn}} \to \|T^m\|^{\frac{1}{m}}$ as $n \to \infty$, it follows that

$$\limsup_n \|T^n\|^{\frac{1}{n}} \leq \|T^m\|^{\frac{1}{m}}.$$

Therefore, since m is an arbitrary positive integer,

$$\limsup_n \|T^n\|^{\frac{1}{n}} \leq \liminf_n \|T^n\|^{\frac{1}{n}},$$

and so $\{\|T^n\|^{\frac{1}{n}}\}$ converges in $\mathbb{R}$. $\square$

For each operator $T \in \mathcal{B}[\mathcal{X}]$ let $r(T)$ denote the limit of $\{\|T^n\|^{\frac{1}{n}}\}$,

$$r(T) = \lim_n \|T^n\|^{\frac{1}{n}}.$$

As we saw in the proof of Lemma 1.10, $r(T) \leq \|T^n\|^{\frac{1}{n}}$ for every $n \geq 1$. In particular, $r(T) \leq \|T\|$. Note that $r(T^k)^{\frac{1}{k}} = \lim_n \|T^{kn}\|^{\frac{1}{kn}} = r(T)$ for each $k \geq 1$ (as $\{\|T^{kn}\|^{\frac{1}{kn}}\}$ is a subsequence of the convergent sequence $\{\|T^n\|^{\frac{1}{n}}\}$). Then $r(T^k) = r(T)^k$ for every positive integer k. Thus, for every operator $T \in \mathcal{B}[\mathcal{X}]$ on a normed space $\mathcal{X}$, and for each nonnegative integer n,

$$0 \leq r(T)^n = r(T^n) \leq \|T^n\| \leq \|T\|^n.$$

An operator $T \in \mathcal{B}[\mathcal{X}]$ on a normed space $\mathcal{X}$ is *normaloid* if $r(T) = \|T\|$. Here is an alternate definition of a normaloid operator.

Theorem 1.11. $r(T) = \|T\|$ *if and only if* $\|T^n\| = \|T\|^n$ *for every* $n \geq 1$.

Proof. Immediate by the above inequalities and the definition of $r(T)$. $\qquad\square$

Theorem 1.12. *Every hyponormal operator is normaloid.*

Proof. Let $T \in \mathcal{B}[\mathcal{H}]$ be a hyponormal operator on a Hilbert space $\mathcal{H}$.

Claim 1. $\|T^n\|^2 \leq \|T^{n+1}\|\|T^{n-1}\|$ for every positive integer n.

Proof. First note that, for any operator $T \in \mathcal{B}[\mathcal{H}]$,

$$\|T^n x\|^2 = \langle T^n x \,;\, T^n x \rangle = \langle T^* T^n x \,;\, T^{n-1} x \rangle \leq \|T^* T^n x\|\|T^{n-1} x\|$$

for each $n \geq 1$ and every $x \in \mathcal{H}$. Now, if T is hyponormal, then

$$\|T^* T^n x\|\|T^{n-1} x\| \leq \|T^{n+1} x\|\|T^{n-1} x\| \leq \|T^{n+1}\|\|T^{n-1}\|\|x\|^2$$

by Theorem 1.7, and hence, for each $n \geq 1$ and every $x \in \mathcal{H}$,

$$\|T^n x\|^2 \leq \|T^{n+1}\|\|T^{n-1}\|\|x\|^2,$$

which ensures the claimed result, thus completing the proof of Claim 1.

Claim 2. $\|T^k\| = \|T\|^k$ for every positive integer $k \leq n$, for all $n \geq 1$.

Proof. The above result holds trivially if $T = O$ and it also holds trivially for $n = 1$ (for all $T \in \mathcal{B}[\mathcal{H}]$). Let $T \neq O$ and suppose the above result holds for some integer $n \geq 1$. By Claim 1 we get

$$\|T\|^{2n} = (\|T\|^n)^2 = \|T^n\|^2 \leq \|T^{n+1}\|\|T^{n-1}\| = \|T^{n+1}\|\|T\|^{n-1}.$$

Therefore, as $\|T^n\| \leq \|T\|^n$ for every $n \geq 1$, and since $T \neq O$,

$$\|T\|^{n+1} = \|T\|^{2n}(\|T\|^{n-1})^{-1} \leq \|T^{n+1}\| \leq \|T\|^{n+1}.$$

Hence $\|T^{n+1}\| = \|T\|^{n+1}$. Then the claimed result holds for $n+1$ whenever it holds for n, which concludes the proof of Claim 2 by induction.

Therefore, $\|T^n\| = \|T\|^n$ for every integer $n \geq 1$ by Claim 2, and so T is normaloid by Theorem 1.11. $\qquad\square$

Since $\|T^{*n}\| = \|T^n\|$ for each $n \geq 1$, it follows that $r(T^*) = r(T)$. Thus an operator T is normaloid if and only if its adjoint T^* is normaloid, and so (by Theorem 1.12) every seminormal operator is normaloid. In particular, *every normal operator is normaloid*. Summing up: an operator T is normal if it commutes with its adjoint (i.e., $TT^* = T^*T$), hyponormal if $TT^* \leq T^*T$, and normaloid if $r(T) = \|T\|$. These classes are related by proper inclusion:

$$\text{Normal} \quad \subset \quad \text{Hyponormal} \quad \subset \quad \text{Normaloid}.$$

1.7 Orthogonal Eigenspaces

Let I denote the identity operator on a Hilbert space $\mathcal{H}$. A *scalar* operator on $\mathcal{H}$ is a scalar multiple of the identity, say λI for any scalar λ in $\mathbb{C}$. Take an arbitrary operator T on $\mathcal{H}$. For every scalar λ in $\mathbb{C}$ consider the kernel $\mathcal{N}(\lambda I - T)$, which is a (closed) subspace of $\mathcal{H}$ (since T is bounded). If this kernel is not zero, then it is called an *eigenspace* of T.

Lemma 1.13. *Take $T \in \mathcal{B}[\mathcal{H}]$ and $\lambda \in \mathbb{C}$ arbitrary.*

(a) *If T is hyponormal, then $\mathcal{N}(\lambda I - T) \subseteq \mathcal{N}(\overline{\lambda} I - T^*)$.*

(b) *If T is normal, then $\mathcal{N}(\lambda I - T) = \mathcal{N}(\overline{\lambda} I - T^*)$.*

Proof. Take any operator $T \in \mathcal{B}[\mathcal{H}]$ and any scalar $\lambda \in \mathbb{C}$. Observe that

$$(\lambda I - T)^*(\lambda I - T) - (\lambda I - T)(\lambda I - T)^* = T^*T - TT^*.$$

Thus $\lambda I - T$ is hyponormal (normal) if and only if T is hyponormal (normal). Therefore, if T is hyponormal, then so is $\lambda I - T$, and hence

$$\|(\overline{\lambda} I - T^*)x\| \ \leq \ \|(\lambda I - T)x\| \ \text{ for every } x \in \mathcal{H} \ \text{ and every } \lambda \in \mathbb{C}$$

by Theorem 1.7, which yields the inclusion in (a). If T is normal, then the preceding inequality becomes an identity, yielding the identity in (b). $\qquad\square$

Lemma 1.14. *Take $T \in \mathcal{B}[\mathcal{H}]$ and $\lambda \in \mathbb{C}$. If $\mathcal{N}(\lambda I - T) \subseteq \mathcal{N}(\overline{\lambda} I - T^*)$, then*

(a) *$\mathcal{N}(\lambda I - T) \perp \mathcal{N}(\nu I - T)$ whenever $\nu \neq \lambda$,   and*

(b) *$\mathcal{N}(\lambda I - T)$ reduces T.*

Proof. (a) Take $x \in \mathcal{N}(\lambda I - T)$ and $y \in \mathcal{N}(\nu I - T)$ arbitrary. Thus $\lambda x = Tx$ and $\nu y = Ty$. If $\mathcal{N}(\lambda I - T) \subseteq \mathcal{N}(\overline{\lambda} I - T^*)$, then $x \in \mathcal{N}(\overline{\lambda} I - T^*)$, and so $\overline{\lambda} x = T^* x$. Then $\langle \nu y \, ; x \rangle = \langle Ty \, ; x \rangle = \langle y \, ; T^* x \rangle = \langle y \, ; \overline{\lambda} x \rangle = \langle \lambda y \, ; x \rangle$, and hence $(\nu - \lambda) \langle y \, ; x \rangle = 0$, which implies that $\langle y \, ; x \rangle = 0$ whenever $\nu \neq \lambda$. (b) If $x \in \mathcal{N}(\lambda I - T) \subseteq \mathcal{N}(\overline{\lambda} I - T^*)$, then $Tx = \lambda x$ and $T^* x = \overline{\lambda} x$, and so $\lambda T^* x = \lambda \overline{\lambda} x = \overline{\lambda} \lambda x = \overline{\lambda} Tx = T \overline{\lambda} x = TT^* x$, which implies that $T^* x \in \mathcal{N}(\lambda I - T)$. Thus $\mathcal{N}(\lambda I - T)$ is T^*-invariant. But $\mathcal{N}(\lambda I - T)$ is T-invariant (if $Tx = \lambda x$, then $T(Tx) = \lambda(Tx)$). Therefore $\mathcal{N}(\lambda I - T)$ reduces T by Lemma 1.6. $\square$

Lemma 1.15. *If* $T \in \mathcal{B}[\mathcal{H}]$ *is hyponormal, then*

(a) $\mathcal{N}(\lambda I - T) \perp \mathcal{N}(\nu I - T)$ *for every* $\lambda, \nu \in \mathbb{C}$ *such that* $\lambda \neq \nu$,   *and*

(b) $\mathcal{N}(\lambda I - T)$ *reduces* T *for every* $\lambda \in \mathbb{C}$.

Proof. Apply Lemma 1.13(a) to Lemma 1.14. $\square$

Theorem 1.16. *If* $\{\lambda_\gamma\}_{\gamma \in \Gamma}$ *is a (nonempty) family of distinct complex numbers, and if* $T \in \mathcal{B}[\mathcal{H}]$ *is hyponormal, then the topological sum*

$$\mathcal{M} = \left(\sum\nolimits_{\gamma \in \Gamma} \mathcal{N}(\lambda_\gamma I - T) \right)^{-}$$

reduces T, *and the restriction of* T *to it,* $T|_{\mathcal{M}} \in \mathcal{B}[\mathcal{M}]$, *is normal.*

Proof. For each $\gamma \in \Gamma \neq \varnothing$ write $\mathcal{N}_\gamma = \mathcal{N}(\lambda_\gamma I - T)$, which is a T-invariant subspace of $\mathcal{H}$. Thus $T|_{\mathcal{N}_\gamma}$ lies in $\mathcal{B}[\mathcal{N}_\gamma]$ and coincides with the scalar operator $\lambda_\gamma I$ on $\mathcal{N}_\gamma$. By Lemma 1.15(a), $\{\mathcal{N}_\gamma\}_{\gamma \in \Gamma}$ is a family of pairwise orthogonal subspaces of $\mathcal{H}$. Take an arbitrary $x \in \left(\sum_{\gamma \in \Gamma} \mathcal{N}_\gamma \right)^{-}$. The Orthogonal Structure Theorem says that $x = \sum_{\gamma \in \Gamma} u_\gamma$ with each u_γ in $\mathcal{N}_\gamma$. Moreover, for each $\gamma \in \Gamma$, Tu_γ and $T^* u_\gamma$ lie in $\mathcal{N}_\gamma$ because $\mathcal{N}_\gamma$ reduces T (Lemmas 1.6 and 1.15(b)). Thus, since T and T^* are linear and continuous, we get $Tx = \sum_{\gamma \in \Gamma} Tu_\gamma$ in $\left(\sum_{\gamma \in \Gamma} \mathcal{N}_\gamma \right)^{-}$ and $T^* x = \sum_{\gamma \in \Gamma} T^* u_\gamma$ in $\left(\sum_{\gamma \in \Gamma} \mathcal{N}_\gamma \right)^{-}$. Then $\left(\sum_{\gamma \in \Gamma} \mathcal{N}_\gamma \right)^{-}$ reduces T (Lemma 1.6). Finally, since $\{\mathcal{N}_\gamma\}_{\gamma \in \Gamma}$ is a family of orthogonal subspaces, it follows by the Orthogonal Structure Theorem that we may identify the topological sum $\mathcal{M}$ with the orthogonal direct sum $\bigoplus_{\gamma \in \Gamma} \mathcal{N}_\gamma$ (cf. Lemma 1.15(a)), where each $\mathcal{N}_\gamma$ reduces $T|_{\mathcal{M}}$ (it reduces T by Lemma 1.15(b)), and so $T|_{\mathcal{M}}$ is identified with the direct sum of operators $\bigoplus_{\gamma \in \Gamma} T|_{\mathcal{N}_\gamma}$, which is normal because each $T|_{\mathcal{N}_\gamma}$ is normal. $\square$

1.8 Compact Operators

A set A in a metric space is *compact* if every open covering of it has a finite subcovering. It is *sequentially compact* if every A-valued sequence has a subsequence that converges in A. It is *totally bounded* if for every $\varepsilon > 0$ there exists a finite partition of it into sets of diameter less than ε. The Compactness Theorem says that *a set in metric space is compact if and only if it is*

sequentially compact, if and only if it is complete and totally bounded. Recall that every compact set is closed, and every totally bounded set is bounded. *On a finite-dimensional normed space a set is compact if and only if it is closed and bounded.* This is the HEINE–BOREL THEOREM.

Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces. A linear transformation $T\colon \mathcal{X} \to \mathcal{Y}$ is *compact* (or *completely continuous*) if it maps bounded subsets of $\mathcal{X}$ into relatively compact subsets of $\mathcal{Y}$. That is, T is compact if $T(A)^-$ is compact in $\mathcal{Y}$ whenever A is bounded in $\mathcal{X}$; equivalently, if $T(A)$ lies in a compact subset of $\mathcal{Y}$ whenever A is bounded in $\mathcal{X}$. *Every linear compact transformation is bounded, thus continuous. Every linear transformation on a finite-dimensional space is compact.* Let $\mathcal{B}_\infty[\mathcal{X}, \mathcal{Y}]$ denote the collection of all compact linear transformations of a normed space $\mathcal{X}$ into a normed space $\mathcal{Y}$ so that $\mathcal{B}_\infty[\mathcal{X}, \mathcal{Y}] \subseteq \mathcal{B}[\mathcal{X}, \mathcal{Y}]$ (and $\mathcal{B}_\infty[\mathcal{X}, \mathcal{Y}] = \mathcal{B}[\mathcal{X}, \mathcal{Y}]$ whenever $\mathcal{X}$ is finite dimensional). In fact, $\mathcal{B}_\infty[\mathcal{X}, \mathcal{Y}]$ is a linear manifold of $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$, which is closed in $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$ if $\mathcal{Y}$ is a Banach space. Set $\mathcal{B}_\infty[\mathcal{X}] = \mathcal{B}_\infty[\mathcal{X}, \mathcal{X}]$ for short, the collection of all *compact operators* on a normed space $\mathcal{X}$. $\mathcal{B}_\infty[\mathcal{X}]$ is an ideal (i.e., a two-sided ideal) of the normed algebra $\mathcal{B}[\mathcal{X}]$. That is, $\mathcal{B}_\infty[\mathcal{X}]$ is a subalgebra of $\mathcal{B}[\mathcal{X}]$ such that the product of a compact operator with a bounded operator is again compact.

We assume that the compact operators act on a complex nonzero Hilbert space $\mathcal{H}$, although the theory for compact operators equally applies (and is usually developed) for operators on Banach spaces. The theorem below says that $\mathcal{R}(\lambda I - T)$ is a subspace of $\mathcal{H}$ if T is compact and λ is nonzero, and the next one says that $\lambda I - T$ is invertible whenever it is injective.

Theorem 1.17. *If $T \in \mathcal{B}_\infty[\mathcal{H}]$ and $\lambda \in \mathbb{C}\backslash\{0\}$, then $\mathcal{R}(\lambda I - T)$ is closed.*

Proof. Let $\mathcal{M}$ be a subspace of a complex Banach space $\mathcal{X}$. Take any compact transformation K in $\mathcal{B}_\infty[\mathcal{M}, \mathcal{X}]$. Let I be the identity on $\mathcal{M}$, let λ be a nonzero complex number, and consider the operator $\lambda I - K$ in $\mathcal{B}[\mathcal{M}, \mathcal{X}]$.

Claim. If $\mathcal{N}(\lambda I - K) = \{0\}$, then $\mathcal{R}(\lambda I - K)$ is closed in $\mathcal{X}$.

Proof. Suppose $\mathcal{N}(\lambda I - K) = \{0\}$. If $\mathcal{R}(\lambda I - K)$ is not closed in $\mathcal{X}$, then $\lambda I - K$ is not bounded below (Theorem 1.2) so that for every $\varepsilon > 0$ there is a nonzero vector $x_\varepsilon \in \mathcal{M}$ such that $\|(\lambda I - K)x_\varepsilon\| < \varepsilon\|x_\varepsilon\|$. Thus there is a sequence $\{x_n\}$ of unit vectors ($\|x_n\| = 1$) in $\mathcal{M}$ for which $\|(\lambda I - K)x_n\| \to 0$. Since K is compact and $\{x_n\}$ is bounded, Proposition 1.S ensures that $\{Kx_n\}$ has a convergent subsequence, say $\{Kx_k\}$, so that $Kx_k \to y \in \mathcal{X}$. However,

$$\|\lambda x_k - y\| = \|\lambda x_k - Kx_k + Kx_k - y\| \leq \|(\lambda I - K)x_k\| + \|Kx_k - y\| \to 0,$$

and so the $\mathcal{M}$-valued sequence $\{\lambda x_k\}$ converges in $\mathcal{X}$ to y. Then $y \in \mathcal{M}$ (because $\mathcal{M}$ is closed in $\mathcal{X}$). Thus, since y is in the domain of K, it follows that $y \in \mathcal{N}(\lambda I - K)$ (because K is continuous, and so $Ky = K\lim_k \lambda x_k = \lambda \lim_k Kx_k = \lambda y$). Moreover, $y \neq 0$ (because $0 \neq |\lambda| = \|\lambda x_k\| \to \|y\|$). Hence $\mathcal{N}(\lambda I - K) \neq \{0\}$, which is a contradiction. Therefore, $\mathcal{R}(\lambda I - K)$ is closed in $\mathcal{X}$, which completes the proof of the claimed result.

Take $T \in \mathcal{B}[\mathcal{H}]$. The restriction $(\lambda I - T)|_{\mathcal{N}(\lambda I - T)^\perp}$ of $\lambda I - T$ to $\mathcal{N}(\lambda I - T)^\perp$ lies in $\mathcal{B}[\mathcal{N}(\lambda I - T)^\perp, \mathcal{H}]$, which is injective (since $\mathcal{N}\big((\lambda I - T)|_{\mathcal{N}(\lambda I - T)^\perp}\big) = \{0\}$) and coincides with $\lambda I - T|_{\mathcal{N}(\lambda I - T)^\perp}$. If T is compact, then so is the restriction $T|_{\mathcal{N}(\lambda I - T)^\perp}$ in $\mathcal{B}[\mathcal{N}(\lambda I - T)^\perp, \mathcal{H}]$ (Proposition 1.V). Thus the above claim says that $(\lambda I - T)|_{\mathcal{N}(\lambda I - T)^\perp} = \lambda I - T|_{\mathcal{N}(\lambda I - T)^\perp}$ (where I in the right-hand side denotes the identity on $\mathcal{N}(\lambda I - T)^\perp$) has a closed range for all $\lambda \neq 0$. But $\mathcal{R}\big((\lambda I - T)|_{\mathcal{N}(\lambda I - T)^\perp}\big) = \mathcal{R}(\lambda I - T)$. $\qquad\square$

Theorem 1.18. *If $T \in \mathcal{B}_\infty[\mathcal{H}]$, $\lambda \in \mathbb{C}\backslash\{0\}$, and $\mathcal{N}(\lambda I - T) = \{0\}$, then $\mathcal{R}(\lambda I - T) = \mathcal{H}$.*

Proof. Take any operator $T \in \mathcal{B}[\mathcal{H}]$ and any scalar $\lambda \in \mathbb{C}$. Consider the sequence $\{\mathcal{M}_n\}_{n=0}^\infty$ of linear manifolds of $\mathcal{H}$ recursively defined by

$$\mathcal{M}_{n+1} = (\lambda I - T)(\mathcal{M}_n) \text{ for every } n \geq 0, \text{ with } \mathcal{M}_0 = \mathcal{H}.$$

It can be verified by induction that

$$\mathcal{M}_{n+1} \subseteq \mathcal{M}_n \text{ for every } n \geq 0.$$

Indeed, $\mathcal{M}_1 = \mathcal{R}(\lambda I - T) \subseteq \mathcal{H} = \mathcal{M}_0$ and, if the above inclusion holds for some $n \geq 0$, then $\mathcal{M}_{n+2} = (\lambda I - T)(\mathcal{M}_{n+1}) \subseteq (\lambda I - T)(\mathcal{M}_n) = \mathcal{M}_{n+1}$, which concludes the induction. Suppose $T \in \mathcal{B}_\infty[\mathcal{H}]$ and $\lambda \neq 0$ so that $\lambda I - T$ has a closed range (Theorem 1.17). If $\mathcal{N}(\lambda I - T) = \{0\}$, then $\lambda I - T$ has a bounded inverse on its range (Theorem 1.2). Since $(\lambda I - T)^{-1}\colon \mathcal{R}(\lambda I - T) \to \mathcal{H}$ is continuous, $\lambda I - T$ sends closed sets into closed sets. (*A map between metric spaces is continuous if and only if the inverse image of closed sets is closed.*) Hence, since $\mathcal{M}_0$ is closed, another induction ensures that

$$\{\mathcal{M}_n\}_{n=0}^\infty \text{ is a decreasing sequence of subspaces of } \mathcal{H}.$$

Now suppose that $\mathcal{R}(\lambda I - T) \neq \mathcal{H}$. If $\mathcal{M}_{n+1} = \mathcal{M}_n$ for some n, then (since $\mathcal{M}_0 = \mathcal{H} \neq \mathcal{R}(\lambda I - T) = \mathcal{M}_1$) there exists an integer $k \geq 1$ such that $\mathcal{M}_{k+1} = \mathcal{M}_k \neq \mathcal{M}_{k-1}$, which leads to a contradiction, namely, if $\mathcal{M}_{k+1} = \mathcal{M}_k$, then $(\lambda I - T)(\mathcal{M}_k) = \mathcal{M}_k$ so that $\mathcal{M}_k = (\lambda I - T)^{-1}(\mathcal{M}_k) = \mathcal{M}_{k-1}$. Outcome: $\mathcal{M}_{n+1}$ is properly included in $\mathcal{M}_n$ for each n; that is,

$$\mathcal{M}_{n+1} \subset \mathcal{M}_n \text{ for every } n \geq 0.$$

Hence each $\mathcal{M}_{n+1}$ is a nonzero proper subspace of the Hilbert space $\mathcal{M}_n$. Then each $\mathcal{M}_{n+1}$ has a nonzero complementary subspace in $\mathcal{M}_n$, namely $\mathcal{M}_n \ominus \mathcal{M}_{n+1} \subset \mathcal{M}_n$. Therefore, for each integer $n \geq 0$ there is an $x_n \in \mathcal{M}_n$ with $\|x_n\| = 1$ such that $\frac{1}{2} < \inf_{u \in \mathcal{M}_{n+1}} \|x_n - u\|$. Thus take any pair of integers $0 \leq m < n$, and set

$$x = x_n + \lambda^{-1}\big((\lambda I - T)x_m - (\lambda I - T)x_n\big)$$

(recall: $\lambda \neq 0$) so that $Tx_n - Tx_m = \lambda(x - x_m)$. Since x lies in $\mathcal{M}_{m+1}$,

$$\tfrac{1}{2}|\lambda| < |\lambda|\|x - x_m\| = \|Tx_n - Tx_m\|,$$

and the sequence $\{Tx_n\}$ has no convergent subsequence (every subsequence of it is not Cauchy). Since $\{x_n\}$ is bounded, it follows that T is not compact (cf. Proposition 1.S), which is a contradiction. Thus $\mathcal{R}(\lambda I - T) = \mathcal{H}$. $\square$

Theorem 1.19. *If* $T \in \mathcal{B}_\infty[\mathcal{H}]$ *and* $\lambda \in \mathbb{C}\backslash\{0\}$ *(i.e., if* T *is compact and* λ *is nonzero), then* $\dim\mathcal{N}(\lambda I - T) = \dim\mathcal{N}(\overline{\lambda}I - T^*) < \infty$.

Proof. Take an arbitrary compact operator $T \in \mathcal{B}_\infty[\mathcal{H}]$ and any nonzero scalar $\lambda \in \mathbb{C}$. Theorem 1.17 says that $\mathcal{R}(\lambda I - T^*) = \mathcal{R}(\lambda I - T^*)^-$, and so $\mathcal{R}(\overline{\lambda}I - T^*) = \mathcal{R}(\overline{\lambda}I - T^*)^-$ by Lemma 1.5. Thus (cf. Lemma 1.4),

$$\mathcal{N}(\lambda I - T) = \{0\} \quad \text{if and only if} \quad \mathcal{R}(\overline{\lambda}I - T^*) = \mathcal{H},$$

$$\mathcal{N}(\overline{\lambda}I - T^*) = \{0\} \quad \text{if and only if} \quad \mathcal{R}(\lambda I - T) = \mathcal{H}.$$

Since $T, T^* \in \mathcal{B}_\infty[\mathcal{H}]$ (cf. Proposition 1.W), we get by Theorem 1.18 that

$$\mathcal{N}(\lambda I - T) = \{0\} \quad \text{implies} \quad \mathcal{R}(\lambda I - T) = \mathcal{H},$$

$$\mathcal{N}(\overline{\lambda}I - T^*) = \{0\} \quad \text{implies} \quad \mathcal{R}(\overline{\lambda}I - T^*) = \mathcal{H}.$$

Hence, $\mathcal{N}(\lambda I - T) = \{0\}$ if and only if $\mathcal{N}(\overline{\lambda}I - T^*) = \{0\}$. Equivalently,

$$\dim\mathcal{N}(\lambda I - T) = 0 \quad \text{if and only if} \quad \dim\mathcal{N}(\overline{\lambda}I - T^*) = 0.$$

Now suppose $\dim\mathcal{N}(\lambda I - T) \neq 0$, and so $\dim\mathcal{N}(\overline{\lambda}I - T^*) \neq 0$. Recall that $\mathcal{N}(\lambda I - T) \neq \{0\}$ is an invariant subspace for T (Proposition 1.F), and that $T|_{\mathcal{N}(\lambda I - T)} = \lambda I$ in $\mathcal{B}[\mathcal{N}(\lambda I - T)]$. If T is compact, then so is $T|_{\mathcal{N}(\lambda I - T)}$ (Proposition 1.V). Then $\lambda I \neq O$ is compact on $\mathcal{N}(\lambda I - T) \neq \{0\}$, and hence $\dim\mathcal{N}(\lambda I - T) < \infty$ (Proposition 1.Y). Dually, since T^* is compact, we get $\dim\mathcal{N}(\overline{\lambda}I - T^*) < \infty$. Thus there are positive integers m and n such that

$$\dim\mathcal{N}(\lambda I - T) = m \quad \text{and} \quad \dim\mathcal{N}(\overline{\lambda}I - T^*) = n.$$

We show next that $m = n$. Let $\{e_i\}_{i=1}^m$ and $\{f_i\}_{i=1}^n$ be orthonormal bases for the Hilbert spaces $\mathcal{N}(\lambda I - T)$ and $\mathcal{N}(\overline{\lambda}I - T^*)$. Set $k = \min\{m, n\} \geq 1$ and consider the mappings $S\colon\mathcal{H} \to \mathcal{H}$ and $S^*\colon\mathcal{H} \to \mathcal{H}$ defined by

$$Sx = \sum\nolimits_{i=1}^k \langle x \,; e_i\rangle f_i \quad \text{and} \quad S^*x = \sum\nolimits_{i=1}^k \overline{\langle x \,; f_i\rangle}e_i$$

for every $x \in \mathcal{H}$. It is clear that S and S^* lie in $\mathcal{B}[\mathcal{H}]$, and also that S^* is the adjoint of S; that is, $\langle Sx \,; y\rangle = \langle x \,; S^*y\rangle$ for every $x, y \in \mathcal{H}$. Actually,

$$\mathcal{R}(S) \subseteq \bigvee\{f_i\}_{i=1}^k \subseteq \mathcal{N}(\overline{\lambda}I - T^*) \quad \text{and} \quad \mathcal{R}(S^*) \subseteq \bigvee\{e_i\}_{i=1}^k \subseteq \mathcal{N}(\lambda I - T)$$

so that S and S^* in $\mathcal{B}[\mathcal{H}]$ are finite-rank operators, thus compact (i.e., they lie in $\mathcal{B}_\infty[\mathcal{H}]$), and hence $T + S$ and $T^* + S^*$ also lie in $\mathcal{B}_\infty[\mathcal{H}]$ (since this is a

subspace of $\mathcal{B}[\mathcal{H}]$). First suppose that $m \leq n$ (which implies that $k = m$). If x lies in $\mathcal{N}(\lambda I - (T + S))$, then $(\lambda I - T)x = Sx$. But $\mathcal{R}(S) \subseteq \mathcal{N}(\overline{\lambda}I - T^*) = \mathcal{R}(\lambda I - T)^\perp$ (Lemma 1.4), and hence $(\lambda I - T)x = Sx = 0$ (since Sx lies in $\mathcal{R}(\lambda I - T)$). Then $x \in \mathcal{N}(\lambda I - T) = \mathrm{span}\{e_i\}_{i=1}^m$ so that $x = \sum_{i=1}^m \alpha_i e_i$ (for some family of scalars $\{\alpha_i\}_{i=1}^m$), and therefore $0 = Sx = \sum_{j=1}^m \alpha_j S e_j = \sum_{j=1}^m \alpha_j \sum_{i=1}^m \langle e_j \, ; e_i \rangle f_i = \sum_{i=1}^m \alpha_i f_i$, so that $\alpha_i = 0$ for every $i = 1, \ldots, m$ — $\{f_i\}_{i=1}^m$ is an orthonormal set, thus linearly independent. That is, $x = 0$. Then $\mathcal{N}(\lambda I - (T + S)) = \{0\}$ and so, by Theorem 1.18,

$$m \leq n \quad \text{implies} \quad \mathcal{R}(\lambda I - (T + S)) = \mathcal{H}.$$

Dually, using exactly the same argument,

$$n \leq m \quad \text{implies} \quad \mathcal{R}(\overline{\lambda}I - (T^* + S^*)) = \mathcal{H}.$$

If $m < n$, then $k = m < m + 1 \leq n$ and $f_{m+1} \in \mathcal{R}(\lambda I - (T + S)) = \mathcal{H}$ so that there exists $v \in \mathcal{H}$ for which $(\lambda I - (T + S))v = f_{m+1}$. Hence

$$
\begin{aligned}
1 = \langle f_{m+1} \, ; f_{m+1} \rangle &= \langle (\lambda I - (T + S))v \, ; f_{m+1} \rangle \\
&= \langle (\lambda I - T)v \, ; f_{m+1} \rangle - \langle Sv \, ; f_{m+1} \rangle = 0,
\end{aligned}
$$

which is a contradiction. Indeed, $\langle (\lambda I - T)v \, ; f_{m+1} \rangle = \langle Sv \, ; f_{m+1} \rangle = 0$ since $f_{m+1} \in \mathcal{N}(\overline{\lambda}I - T^*) = \mathcal{R}(\lambda I - T)^\perp$ and $Sv \in \mathcal{R}(S) \subseteq \mathrm{span}\{f_i\}_{i=1}^m$. If $n < m$, then $k = n < n + 1 \leq m$, and $e_{n+1} \in \mathcal{R}(\overline{\lambda}I - (T^* + S^*)) = \mathcal{H}$ so that there exists $u \in \mathcal{H}$ for which $(\overline{\lambda}I - (T^* + S^*))u = e_{n+1}$. Hence

$$
\begin{aligned}
1 = \langle e_{m+1} \, ; e_{m+1} \rangle &= \langle (\overline{\lambda}I - (T^* + S^*))u \, ; e_{n+1} \rangle \\
&= \langle (\overline{\lambda}I - T^*)u \, ; e_{n+1} \rangle - \langle S^* u \, ; e_{n+1} \rangle = 0,
\end{aligned}
$$

which is again a contradiction (because $e_{n+1} \in \mathcal{N}(\lambda I - T) = \mathcal{R}(\overline{\lambda}I - T^*)^\perp$ and $S^* u \in \mathcal{R}(S^*) \subseteq \mathrm{span}\{e_i\}_{i=1}^n$). Therefore, $m = n$. $\quad\square$

Together, the results of Theorems 1.17 and 1.19 are referred to as the Fredholm Alternative, which is stated below.

Corollary 1.20. (FREDHOLM ALTERNATIVE). *If $T \in \mathcal{B}_\infty[\mathcal{H}]$ and $\lambda \in \mathbb{C}\backslash\{0\}$, then $\mathcal{R}(\lambda I - T)$ is closed and $\dim \mathcal{N}(\lambda I - T) = \dim \mathcal{N}(\overline{\lambda}I - T^*) < \infty$.*

1.9 Additional Propositions

Let $\mathcal{X}$ be a complex linear space. $f \colon \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ is a sesquilinear form if it is additive in both arguments, homogeneous in the first argument, and conjugate homogeneous in the second argument. A Hermitian symmetric sesquilinear form that induces a positive quadratic form is an inner product.

Proposition 1.A. (POLARIZATION IDENTITIES). *If $f \colon \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ is a sesquilinear form on a complex linear space $\mathcal{X}$, then, for every $x, y \in \mathcal{X}$,*

(a$_1$) $$f(x, y) = \tfrac{1}{4}\big(f(x + y, x + y) - f(x - y, x - y)$$
$$+ \, i \, f(x + iy, x + iy) - i \, f(x - iy, x - iy)\big).$$

If $\mathcal{X}$ is a complex inner product space and $S, T \in \mathcal{B}[\mathcal{X}]$, then, for $x, y \in \mathcal{X}$,

(a$_2$) $$\langle Sx \, ; Ty \rangle = \tfrac{1}{4}\big(\langle S(x + y) \, ; T(x + y)\rangle - \langle S(x - y) \, ; T(x - y)\rangle$$
$$+ \, i\langle S(x + iy) \, ; T(x + iy)\rangle - i\langle S(x - iy) \, ; T(x - iy)\rangle\big).$$

(PARALLELOGRAM LAW). *If $\mathcal{X}$ is an inner product space and $\|\cdot\|$ is the norm generated by the inner product, then, for every $x, y \in \mathcal{X}$,*

(b) $$\|x + y\|^2 + \|x - y\|^2 = 2\big(\|x\|^2 + \|y\|^2\big).$$

Proposition 1.B. *A linear manifold of a Banach space is a Banach space if and only if it is a subspace. A linear transformation is bounded if and only if it maps bounded sets into bounded sets. If $\mathcal{X}$ and $\mathcal{Y}$ are nonzero normed spaces, then $\mathcal{B}[\mathcal{X}, \mathcal{Y}]$ is a Banach space if and only if $\mathcal{Y}$ is a Banach space.*

Recall that the sum of two subspaces of a Hilbert space may not be a subspace (i.e., the sum may not be closed if the subspaces are not orthogonal).

Proposition 1.C. *If $\mathcal{M}$ and $\mathcal{N}$ are subspaces of a normed space $\mathcal{X}$, and if $\dim \mathcal{N} < \infty$, then the sum $\mathcal{M} + \mathcal{N}$ is a subspace of $\mathcal{X}$.*

Every subspace of a Hilbert has a complementary subspace (e.g., its orthogonal complement). However, this is not true for every Banach space.

Proposition 1.D. *If $\mathcal{M}$ and $\mathcal{N}$ are complementary subspaces of a Banach space $\mathcal{X}$, then the unique projection $E : \mathcal{X} \to \mathcal{X}$ with $\mathcal{R}(E) = \mathcal{M}$ and $\mathcal{N}(E) = \mathcal{N}$ is continuous. Conversely, if $E \in \mathcal{B}[\mathcal{X}]$ is a projection, then $\mathcal{R}(E)$ and $\mathcal{N}(E)$ are complementary subspaces of $\mathcal{X}$.*

Proposition 1.E. *Let $\mathcal{X}$ be a normed space. The subspaces $\mathcal{N}(T)$ and $\mathcal{R}(T)^-$ are invariant for any operator $T \in \mathcal{B}[\mathcal{X}]$. Take another operator $S \in \mathcal{B}[\mathcal{X}]$. If S and T commute, then the subspaces $\mathcal{N}(S)$, $\mathcal{N}(T)$, $\mathcal{R}(S)^-$, and $\mathcal{R}(T)^-$ are invariant for both S and T.*

Proposition 1.F. *If T is an operator on a normed space, then $\mathcal{N}(p(T))$ and $\mathcal{R}(p(T))^-$ are T-invariant subspaces for every polynomial $p(T)$ of T — in particular, $\mathcal{N}(\lambda I - T)$ and $\mathcal{R}(\lambda I - T)^-$ are T-invariant for every $\lambda \in \mathbb{C}$.*

Proposition 1.G. *Let $\mathcal{M}$ be a linear manifold of a normed space $\mathcal{X}$, let $\mathcal{Y}$ be Banach space, and take $T \in \mathcal{B}[\mathcal{M}, \mathcal{Y}]$.*

(a) *There exists a unique extension $\widehat{T} \in \mathcal{B}[\mathcal{M}^-, \mathcal{Y}]$ of $T \in \mathcal{B}[\mathcal{M}, \mathcal{Y}]$ over the closure of $\mathcal{M}$. Moreover $\|\widehat{T}\| = \|T\|$.*

(b) *If $\mathcal{X}$ is a Hilbert space, then $\widehat{T}E \in \mathcal{B}[\mathcal{X}, \mathcal{Y}]$ is a (bounded linear) extension of $T \in \mathcal{B}[\mathcal{M}, \mathcal{Y}]$ over the whole space $\mathcal{X}$, where $E \in \mathcal{B}[\mathcal{X}]$ is the orthogonal projection on the subspace $\mathcal{M}^-$ of $\mathcal{X}$ so that $\|\widehat{T}E\| \leq \|T\|$.*

Proposition 1.H. *Let $\mathcal{M}$ and $\mathcal{N}$ be subspaces of a Hilbert space.*

(a) $\mathcal{M}^\perp \cap \mathcal{N}^\perp = (\mathcal{M} + \mathcal{N})^\perp$.

(b) $\dim \mathcal{N} < \infty \implies \dim \left(\mathcal{N} \ominus (\mathcal{M} \cap \mathcal{N}) \right) = \dim \left(\mathcal{M}^\perp \ominus (\mathcal{M} + \mathcal{N})^\perp \right)$.

Proposition 1.I. *A subspace $\mathcal{M}$ of a Hilbert space $\mathcal{H}$ is invariant (or reducing) for a nonzero $T \in \mathcal{B}[\mathcal{H}]$ if and only if the unique orthogonal projection $E \in \mathcal{B}[\mathcal{H}]$ with $\mathcal{R}(E) = \mathcal{M}$ is such that $ETE = TE$ (or $TE = ET$).*

Take $T \in \mathcal{B}[\mathcal{H}]$ and $S \in \mathcal{B}[\mathcal{K}]$ on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, and $X \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$ of $\mathcal{H}$ into $\mathcal{K}$. If $XT = SX$, then X *intertwines* T to S (or T is *intertwined* to S through X). If there is an X with dense range intertwining T to S, then T is *densely intertwined* to S. If X has dense range and is injective, then it is *quasiinvertible* (or a *quasiaffinity*). If a quasiinvertible X intertwines T to S, then T is a *quasiaffine transform* of S. If T is a quasiaffine transform of S and S is a quasiaffine transform of T, then T and S are *quasisimilar*. If an invertible X (with a bounded inverse) intertwines T to S (so that X^{-1} intertwines S to T), then T and S are *similar* (or *equivalent*). Unitary equivalence is the special case of similarity through a (surjective) isometry: operators T and S are *unitarily equivalent* (notation: $T \cong S$) if there exists a unitary transformation X intertwining them.

A linear manifold (or a subspace) of normed space $\mathcal{X}$ is *hyperinvariant* for an operator $T \in \mathcal{B}[\mathcal{X}]$ if it is invariant for every operator in $\mathcal{B}[\mathcal{X}]$ that commutes with T. Obviously, hyperinvariant subspaces are invariant.

Proposition 1.J. *Similarity preserves invariant subspaces (if two operators are similar, and if one has a nontrivial invariant subspace, then so has the other), and quasisimilarity preserves hyperinvariant subspaces (if two operators are quasisimilar, and if one has a nontrivial hyperinvariant subspace, then so has the other).*

Proposition 1.K. *If $\{E_k\}$ is an orthogonal sequence of orthogonal projections on a Hilbert space $\mathcal{H}$, then $\sum_{k=1}^n E_k \xrightarrow{s} E$, where $E \in \mathcal{B}[\mathcal{H}]$ is the orthogonal projection with $\mathcal{R}(E) = \left(\sum_{k \in \mathbb{N}} \mathcal{R}(E_k) \right)^-$.*

Proposition 1.L. *A bounded linear transformation between Hilbert spaces is invertible if and only if its adjoint is (i.e., $T \in \mathcal{G}[\mathcal{H}, \mathcal{K}] \iff T^* \in \mathcal{G}[\mathcal{K}, \mathcal{H}]$). Moreover, $(T^*)^{-1} = (T^{-1})^*$.*

Proposition 1.M. *Every nonnegative operator $Q \in \mathcal{B}[\mathcal{H}]$ on a Hilbert space $\mathcal{H}$ has a unique nonnegative square root $Q^{\frac{1}{2}} \in \mathcal{B}[\mathcal{H}]$ (i.e., $(Q^{\frac{1}{2}})^2 = Q$), which commutes with every operator in $\mathcal{B}[\mathcal{H}]$ that commutes with Q.*

Proposition 1.N. *Every $T \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$ between Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ has a unique polar decomposition $T = W|T|$, where $W \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$ is a partial isometry (i.e., $W|_{N(W)^\perp} : \mathcal{N}(W)^\perp \to \mathcal{K}$ is an isometry) and $|T| = (T^*T)^{\frac{1}{2}}$.*

Proposition 1.O. *Every operator $T \in \mathcal{B}[\mathcal{H}]$ on a complex Hilbert space $\mathcal{H}$ has a unique Cartesian decomposition $T = A + iB$, where $A = \frac{1}{2}(T + T^*)$ and $B = -\frac{i}{2}(T - T^*)$ are self-adjoint operators on $\mathcal{H}$. Moreover, T is normal if and only if A and B commute and, in this case, $T^*T = A^2 + B^2$ and $\max\{\|A\|^2, \|B\|^2\} \le \|T\|^2 \le \|A\|^2 + \|B\|^2$.*

Proposition 1.P. *The restriction $T|_\mathcal{M}$ of a hyponormal operator T to an invariant subspace $\mathcal{M}$ is hyponormal. If $T|_\mathcal{M}$ is normal, then $\mathcal{M}$ reduces T.*

Proposition 1.Q. *The restriction $T|_\mathcal{M}$ of a normal operator T to an invariant subspace $\mathcal{M}$ is normal if and only if $\mathcal{M}$ reduces T.*

Proposition 1.R. *A transformation U between Hilbert spaces is unitary if and only if it is an invertible contraction whose inverse also is a contraction ($\|U\| \le 1$ and $\|U^{-1}\| \le 1$), which happens if and only if $\|U\| = \|U^{-1}\| = 1$.*

Proposition 1.S. *A linear transformation between normed spaces is compact if and only if it maps bounded sequences into sequences that have a convergent subsequence.*

Proposition 1.T. *A linear transformation of a normed space into a Banach space is compact if and only if it maps bounded sets into totally bounded sets.*

Proposition 1.U. *A linear transformation of a Hilbert space into a normed space is compact if and only if it maps weakly convergent sequences into strongly convergent sequences.*

Proposition 1.V. *The restriction of a compact linear transformation to a linear manifold is again a compact linear transformation.*

Proposition 1.W. *A linear transformation between Hilbert spaces is compact if and only if its adjoint is (i.e., $T \in \mathcal{B}_\infty[\mathcal{H}, \mathcal{K}] \iff T^* \in \mathcal{B}_\infty[\mathcal{K}, \mathcal{H}]$).*

Proposition 1.X. *A finite-rank (i.e., one with a finite-dimensional range) bounded linear transformation between normed spaces is compact.*

Proposition 1.Y. *The identity I on a normed space $\mathcal{X}$ is compact if and only if $\dim(\mathcal{X}) < \infty$. Hence, if $T \in \mathcal{B}_\infty[\mathcal{X}]$ is invertible, then $\dim(\mathcal{X}) < \infty$.*

Proposition 1.Z. *If a sequence of finite-rank bounded linear transformations between Banach spaces converges uniformly, then its limit is compact.*

Notes: These are standard results that will be required in the sequel. Proofs can be found in many texts on operator theory (see Suggested Reading). For

instance, Proposition 1.A can be found in [66, Propositions 5.4 and Problem 5.3], and Proposition 1.B in [66, Propositions 4.7, 4.12, 4.15, and Example 4.P]. Proposition 1.C can be found in [27, Proposition III.4.3] (for a Hilbert space proof see [50, Problem 13]), and Proposition 1.D in [27, Theorem III.13.2] or [66, Problem 4.35]. For Propositions 1.E and 1.F see [66, Problems 4.20 and 4.22]. Proposition 1.G(a) is referred to as *extension by continuity*; see, e.g., [66, Theorem 4.35]. It implies Proposition 1.G(b): an orthogonal projection is continuous, and $\widehat{T} = \widehat{T}E|_{\mathcal{R}(E)} \in \mathcal{B}[\mathcal{M}^-, \mathcal{H}]$ is a restriction of $\widehat{T}E \in \mathcal{B}[\mathcal{X}, \mathcal{H}]$ to $\mathcal{R}(E) = \mathcal{M}^-$. Proposition 1.H will be needed in Theorem 5.4. For Proposition 1.H(a) see [66, Problems 5.8]. A proof of Proposition 1.H(b) goes as follows.

Proof of Proposition 1.H(b). Since $(\mathcal{M} \cap \mathcal{N}) \perp (\mathcal{M}^\perp \cap \mathcal{N})$, consider the orthogonal projection $E \in \mathcal{B}[\mathcal{H}]$ on $\mathcal{M}^\perp \cap \mathcal{N}$ (i.e., $\mathcal{R}(E) = \mathcal{M}^\perp \cap \mathcal{N}$) along $\mathcal{M} \cap \mathcal{N}$ (i.e., $\mathcal{N}(E) = \mathcal{M} \cap \mathcal{N}$). Let $L : \mathcal{N} \ominus (\mathcal{M} \cap \mathcal{N}) \to \mathcal{M}^\perp \cap \mathcal{N}$ be the restriction of E to $\mathcal{N} \ominus (\mathcal{M} \cap \mathcal{N})$. It is clear that $L = E|_{\mathcal{N} \ominus (\mathcal{M} \cap \mathcal{N})}$ is linear, injective ($\mathcal{N}(L) = \{0\}$), and surjective ($\mathcal{R}(L) = \mathcal{R}(E) = \mathcal{M}^\perp \cap \mathcal{N}$). Therefore, $\mathcal{N} \ominus (\mathcal{M} \cap \mathcal{N})$ is isomorphic to $\mathcal{M}^\perp \cap \mathcal{N}$. Since $\mathcal{M}$ is closed and $\mathcal{N}$ is finite dimensional, $\mathcal{M} + \mathcal{N}$ is closed (even if $\mathcal{M}$ and $\mathcal{N}$ are not orthogonal — cf. Proposition 1.C). Then (Projection Theorem) $\mathcal{H} = (\mathcal{M} + \mathcal{N})^\perp \oplus (\mathcal{M} + \mathcal{N})$ and $\mathcal{M}^\perp = (\mathcal{M} + \mathcal{N})^\perp \oplus [\mathcal{M}^\perp \ominus (\mathcal{M} + \mathcal{N})^\perp]$, and hence $\mathcal{M}^\perp \ominus (\mathcal{M} + \mathcal{N})^\perp$ coincides with $\mathcal{M}^\perp \cap (\mathcal{M} + \mathcal{N}) = \mathcal{M}^\perp \cap \mathcal{N}$. Therefore, $\mathcal{N} \ominus (\mathcal{M} \cap \mathcal{N})$ is isomorphic to $\mathcal{M}^\perp \ominus (\mathcal{M} + \mathcal{N})^\perp$, and so they have the same dimension. $\square$

For Proposition 1.I see, e.g., [63, Solutions 4.1 and 4.6], and for Proposition 1.J see, e.g., [62, Corollaries 4.2 and 4.8]. By the way, *does quasisimilarity preserve nontrivial invariant subspaces*? (See, e.g., [76, p. 194].) As for Propositions 1.K and 1.L see [66, Propositions 5.28 and p. 385]). Propositions 1.M, 1.N, and 1.O are the classical Square Root Theorem, the Polar Decomposition Theorem, and the Cartesian Decomposition Theorem (see, e.g., [66, Theorems 5.85 and 5.89, and Problem 5.46]). Propositions 1.P, 1.Q, 1.R can be found in [66, Problems 6.16 and 6.17, and Theorem 5.73]. Propositions 1.S, 1.T, 1.U, 1.V, 1.W, 1.X, 1.Y, and 1.Z are all related to compact operators (see, e.g., [66, Theorem 4.52(d,e), Problem 4.51, p. 258, Problem 5.41, Proposition 4.50, Remark to Theorem 4.52, Corollary 4.55]).

Suggested Reading

Akhiezer and Glazman [3, 4]
Beals [12]
Berberian [17, 18]
Brown and Pearcy [21, 22]
Conway [27, 28, 29]
Douglas [34]
Dunford and Schwartz [39]

Fillmore [42]
Halmos [47, 48, 50]
Harte [52]
Istrăţescu [58]
Kubrusly [62, 63, 66]
Radjavi and Rosenthal [76]
Weidmann [89]

2

Spectrum

Let $T\colon \mathcal{D}(T) \to \mathcal{X}$ be a linear transformation, where $\mathcal{X}$ is a nonzero normed space and $\mathcal{D}(T)$, the domain of T, is a linear manifold of $\mathcal{X}$. The general notion of spectrum, which applies to bounded or unbounded transformations, goes as follows. Let $\mathbb{F}$ denote either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$, and let I be the identity on $\mathcal{X}$. The *resolvent set* $\rho(T)$ of T is the set of all scalars λ in $\mathbb{F}$ for which the linear transformation $\lambda I - T\colon \mathcal{D}(T) \to \mathcal{X}$ has a densely defined continuous inverse. That is,

$$\rho(T) = \big\{\lambda \in \mathbb{F}\colon\ (\lambda I - T)^{-1} \in \mathcal{B}[\mathcal{R}(\lambda I - T), \mathcal{D}(T)] \ \text{ and } \ \mathcal{R}(\lambda I - T)^- = \mathcal{X}\big\}$$

(see, e.g., [8, Definition 18.2] — there are different definitions of the resolvent set for unbounded linear transformations, but they all coincide for the bounded case). The *spectrum* $\sigma(T)$ of T is the complement of set $\rho(T)$ in $\mathbb{F}$. We shall however restrict the theory to operators on a complex Banach space (i.e., to bounded linear transformations of a complex Banach space to itself).

2.1 Basic Spectral Properties

Throughout this chapter $T\colon \mathcal{X} \to \mathcal{X}$ will be a *bounded* linear transformation of $\mathcal{X}$ into itself (i.e, an operator on $\mathcal{X}$), so that $\mathcal{D}(T) = \mathcal{X}$, where $\mathcal{X} \neq \{0\}$ is a *complex Banach space*. That is, $T \in \mathcal{B}[\mathcal{X}]$, where $\mathcal{X}$ is a nonzero complex Banach space. In such a case (i.e., in the *unital complex Banach algebra* $\mathcal{B}[\mathcal{X}]$), Theorem 1.2 ensures that the resolvent set $\rho(T)$ is precisely the set of all complex numbers λ for which $\lambda I - T \in \mathcal{B}[\mathcal{X}]$ is invertible (i.e., has a bounded inverse on $\mathcal{X}$). Therefore (cf. Theorem 1.1),

$$\begin{aligned}
\rho(T) &= \big\{\lambda \in \mathbb{C}\colon\ \lambda I - T \in \mathcal{G}[\mathcal{X}]\big\} \\
&= \big\{\lambda \in \mathbb{C}\colon\ \lambda I - T \text{ has an inverse in } \mathcal{B}[\mathcal{X}]\big\} \\
&= \big\{\lambda \in \mathbb{C}\colon\ \mathcal{N}(\lambda I - T) = \{0\} \ \text{ and } \ \mathcal{R}(\lambda I - T) = \mathcal{X}\big\},
\end{aligned}$$

and so

$$\begin{aligned}
\sigma(T) = \mathbb{C}\setminus\rho(T) &= \big\{\lambda \in \mathbb{C}\colon\ \lambda I - T \text{ has no inverse in } \mathcal{B}[\mathcal{X}]\big\} \\
&= \big\{\lambda \in \mathbb{C}\colon\ \mathcal{N}(\lambda I - T) \neq \{0\} \ \text{ or } \ \mathcal{R}(\lambda I - T) \neq \mathcal{X}\big\}.
\end{aligned}$$

Theorem 2.1. *The resolvent set $\rho(T)$ is nonempty and open, and the spectrum $\sigma(T)$ is compact.*

Proof. Take any $T \in \mathcal{B}[\mathcal{X}]$. By the Neumann expansion (Theorem 1.3), if $\|T\| < |\lambda|$, then $\lambda \in \rho(T)$. Equivalently, since $\sigma(T) = \mathbb{C}\backslash\rho(T)$,

$$|\lambda| \leq \|T\| \quad \text{for every } \lambda \in \sigma(T).$$

Thus $\sigma(T)$ is bounded, and therefore $\rho(T) \neq \varnothing$.

Claim. If $\lambda \in \rho(T)$, then the open ball $B_\delta(\lambda)$ with center at λ and (positive) radius $\delta = \|(\lambda I - T)^{-1}\|^{-1}$ is included in $\rho(T)$.

Proof. If $\lambda \in \rho(T)$, then $\lambda I - T \in \mathcal{G}[\mathcal{X}]$ so that $(\lambda I - T)^{-1}$ is nonzero and bounded, and hence $0 < \|(\lambda I - T)^{-1}\|^{-1} < \infty$. Set $\delta = \|(\lambda I - T)^{-1}\|^{-1}$, let $B_\delta(0)$ be the nonempty open ball of radius δ about the origin of the complex plane $\mathbb{C}$, and take any ν in $B_\delta(0)$. Since $|\nu| < \|(\lambda I - T)^{-1}\|^{-1}$, it follows that $\|\nu(\lambda I - T)^{-1}\| < 1$. Then $[I - \nu(\lambda I - T)^{-1}] \in \mathcal{G}[\mathcal{X}]$ by Theorem 1.3, and so $(\lambda - \nu)I - T = (\lambda I - T)[I - \nu(\lambda I - T)^{-1}] \in \mathcal{G}[\mathcal{X}]$. Thus $\lambda - \nu \in \rho(T)$ so that

$$B_\delta(\lambda) = B_\delta(0) + \lambda = \big\{\nu \in \mathbb{C}: \ \nu = \nu + \lambda \text{ for some } \nu \in B_\delta(0)\big\} \subseteq \rho(T),$$

which completes the proof of the claimed result.

Thus $\rho(T)$ is open (it includes a nonempty open ball centered at each of its points) and so $\sigma(T)$ is closed. Compact in $\mathbb{C}$ means closed and bounded. $\square$

Remark. Since $B_\delta(\lambda) \subseteq \rho(T)$, it follows that the distance of any λ in $\rho(T)$ to the spectrum $\sigma(T)$ is greater than δ; that is (compare with Proposition 2.E),

$$\lambda \in \rho(T) \quad \text{implies} \quad \|(\lambda I - T)^{-1}\|^{-1} \leq d(\lambda, \sigma(T)).$$

The *resolvent function* $R_T \colon \rho(T) \to \mathcal{G}[\mathcal{X}]$ of an operator $T \in \mathcal{B}[\mathcal{X}]$ is the mapping of the resolvent set $\rho(T)$ of T into the group $\mathcal{G}[\mathcal{X}]$ of all invertible operators from $\mathcal{B}[\mathcal{X}]$ defined by

$$R_T(\lambda) = (\lambda I - T)^{-1} \quad \text{for every } \lambda \in \rho(T).$$

Since $R_T(\lambda) - R_T(\nu) = R_T(\lambda)[R_T(\nu)^{-1} - R_T(\lambda)^{-1}]R_T(\nu)$, it follows that

$$R_T(\lambda) - R_T(\nu) = (\nu - \lambda)R_T(\lambda)R_T(\nu)$$

for every $\lambda, \nu \in \rho(T)$ (because $R_T(\nu)^{-1} - R_T(\lambda)^{-1} = (\nu - \lambda)I$). This is the *resolvent identity*. Swapping λ and ν in the resolvent identity, it follows that $R_T(\lambda)$ and $R_T(\nu)$ commute for every $\lambda, \nu \in \rho(T)$. Also, $TR_T(\lambda) = R_T(\lambda)T$ for every $\lambda \in \rho(T)$ (since $R_T(\lambda)^{-1}R_T(\lambda) = R_T(\lambda)R_T(\lambda)^{-1}$ trivially).

Let Λ be a nonempty open subset of the complex plane $\mathbb{C}$. Take a function $f \colon \Lambda \to \mathbb{C}$ and a point $\nu \in \Lambda$. Suppose there exists a complex number

$f'(\nu)$ with the following property. For every $\varepsilon > 0$ there is a $\delta > 0$ such that $\left|\frac{f(\lambda)-f(\nu)}{\lambda-\nu} - f'(\nu)\right| < \varepsilon$ for all λ in Λ for which $0 < |\lambda - \nu| < \delta$. If there exists such an $f'(\nu) \in \mathbb{C}$, then it is called the *derivative* of f at ν. If $f'(\nu)$ exists for every ν in Λ, then $f : \Lambda \to \mathbb{C}$ is *analytic* (or *holomorphic*) on Λ. A function $f : \mathbb{C} \to \mathbb{C}$ is *entire* if it is analytic on the whole complex plane $\mathbb{C}$. To prove the next result we need the LIOUVILLE THEOREM, which says that *every bounded entire function is constant*.

Theorem 2.2. *If $\mathcal{X}$ is nonzero, then the spectrum $\sigma(T)$ is nonempty.*

Proof. Let $T \in \mathcal{B}[\mathcal{X}]$ be an operator on a nonzero complex Banach space $\mathcal{X}$, and let $\mathcal{B}[\mathcal{X}]^*$ stand for the dual of $\mathcal{B}[\mathcal{X}]$. That is, $\mathcal{B}[\mathcal{X}]^* = \mathcal{B}[\mathcal{B}[\mathcal{X}], \mathbb{C}]$ is the Banach space of all bounded linear functionals on $\mathcal{B}[\mathcal{X}]$. Since $\mathcal{X} \neq \{0\}$, it follows that $\mathcal{B}[\mathcal{X}] \neq \{O\}$, and so $\mathcal{B}[\mathcal{X}]^* \neq \{0\}$, which is a consequence of the Hahn–Banach Theorem (see, e.g., [66, Corollary 4.64]). Take any nonzero η in $\mathcal{B}[\mathcal{X}]^*$ (i.e., a nonzero bounded linear functional $\eta : \mathcal{B}[\mathcal{X}] \to \mathbb{C}$), and consider the composition of it with the resolvent function, $\eta \circ R_T : \rho(T) \to \mathbb{C}$. Recall that $\rho(T) = \mathbb{C} \backslash \sigma(T)$ is nonempty and open in $\mathbb{C}$.

Claim 1. If $\sigma(T)$ is empty, then $\eta \circ R_T : \rho(T) \to \mathbb{C}$ is bounded.

Proof. The resolvent function $R_T : \rho(T) \to \mathcal{G}[\mathcal{X}]$ is continuous (since scalar multiplication and addition are continuous mappings, and inversion is also a continuous mapping; see, e.g., [66, Problem 4.48]). Thus $\|R_T(\cdot)\| : \rho(T) \to \mathbb{R}$ is continuous. Then $\sup_{|\lambda| \leq \|T\|} \|R_T(\lambda)\| < \infty$ if $\sigma(T) = \varnothing$. Indeed, if $\sigma(T)$ is empty, then $\rho(T) \cap B_{\|T\|}[0] = B_{\|T\|}[0] = \{\lambda \in \mathbb{C} : |\lambda| \leq \|T\|\}$ is a compact set in $\mathbb{C}$, so that the continuous function $\|R_T(\cdot)\|$ attains its maximum on it by the WEIERSTRASS THEOREM: *a continuous real-valued function attains its maximum and minimum on any compact set in a metric space*. On the other hand, if $\|T\| < |\lambda|$, then $\|R_T(\lambda)\| \leq (|\lambda| - \|T\|)^{-1}$ (cf. Remark that follows Theorem 1.3), so that $\|R_T(\lambda)\| \to 0$ as $|\lambda| \to \infty$. Thus, since $\|R_T(\cdot)\| : \rho(T) \to \mathbb{R}$ is continuous, $\sup_{\|T\| \leq \lambda} \|R_T(\lambda)\| < \infty$. Hence $\sup_{\lambda \in \rho(T)} \|R_T(\lambda)\| < \infty$, and so

$$\sup_{\lambda \in \rho(T)} \|(\eta \circ R_T)(\lambda)\| \leq \|\eta\| \sup_{\lambda \in \rho(T)} \|R_T(\lambda)\| < \infty,$$

which completes the proof of Claim 1.

Claim 2. $\eta \circ R_T : \rho(T) \to \mathbb{C}$ is analytic.

Proof. If λ and ν are distinct points in $\rho(T)$, then

$$\frac{R_T(\lambda) - R_T(\nu)}{\lambda - \nu} + R_T(\nu)^2 = \big(R_T(\nu) - R_T(\lambda)\big) R_T(\nu)$$

by the resolvent identity. Set $f = \eta \circ R_T : \rho(T) \to \mathbb{C}$. Let $f' : \rho(T) \to \mathbb{C}$ be defined by $f'(\lambda) = -\eta(R_T(\lambda)^2)$ for each $\lambda \in \rho(T)$. Therefore,

$$\left|\frac{f(\lambda) - f(\nu)}{\lambda - \nu} - f'(\nu)\right| = \big|\eta\big[\big(R_T(\nu) - R_T(\lambda)\big) R_T(\nu)\big]\big|$$

$$\leq \|\eta\| \|R_T(\nu)\| \|R_T(\nu) - R_T(\lambda)\|$$

so that $f \colon \rho(T) \to \mathbb{C}$ is analytic because $R_T \colon \rho(T) \to \mathcal{G}[\mathcal{X}]$ is continuous, which completes the proof of Claim 2.

Thus, by Claims 1 and 2, if $\sigma(T) = \varnothing$ (i.e., if $\rho(T) = \mathbb{C}$), then $\eta \circ R_T \colon \mathbb{C} \to \mathbb{C}$ is a bounded entire function, and so a constant function by the Liouville Theorem. But (see proof of Claim 1) $\|R_T(\lambda)\| \to 0$ as $|\lambda| \to \infty$, and hence $\eta(R_T(\lambda)) \to 0$ as $|\lambda| \to \infty$ (since η is continuous). Then $\eta \circ R_T = 0$ for all η in $\mathcal{B}[\mathcal{H}]^* \neq \{0\}$ so that $R_T = O$ (by the Hahn–Banach Theorem). That is, $(\lambda I - T)^{-1} = O$ for $\lambda \in \mathbb{C}$, which is a contradiction. Thus $\sigma(T) \neq \varnothing$. $\square$

Remark. $\sigma(T)$ is compact and nonempty, and so is its boundary $\partial\sigma(T)$. Hence, $\partial\sigma(T) = \partial\rho(T) \neq \varnothing$.

2.2 A Classical Partition of the Spectrum

The spectrum $\sigma(T)$ of an operator T in $\mathcal{B}[\mathcal{X}]$ is the set of all scalars λ in $\mathbb{C}$ for which the operator $\lambda I - T$ fails to be an invertible element of the algebra $\mathcal{B}[\mathcal{X}]$ (i.e., fails to have a bounded inverse on $\mathcal{R}(\lambda I - T) = \mathcal{X}$). According to the nature of such a failure, $\sigma(T)$ can be split into many disjoint parts. A classical partition comprises three parts. The set $\sigma_P(T)$ of those λ for which $\lambda I - T$ has no inverse (i.e., such that the operator $\lambda I - T$ is not injective) is the *point spectrum* of T,

$$\sigma_P(T) = \big\{\lambda \in \mathbb{C} \colon \ \mathcal{N}(\lambda I - T) \neq \{0\}\big\}.$$

A scalar $\lambda \in \mathbb{C}$ is an *eigenvalue* of T if there exists a nonzero vector x in $\mathcal{X}$ such that $Tx = \lambda x$. Equivalently, λ is an *eigenvalue* of T if $\mathcal{N}(\lambda I - T) \neq \{0\}$. If $\lambda \in \mathbb{C}$ is an eigenvalue of T, then the nonzero vectors in $\mathcal{N}(\lambda I - T)$ are the *eigenvectors* of T, and $\mathcal{N}(\lambda I - T)$ is the *eigenspace* (which is a subspace of $\mathcal{X}$), associated with the eigenvalue λ. The *multiplicity* of an eigenvalue is the dimension of the respective eigenspace. Thus *the point spectrum of T is precisely the set of all eigenvalues of T*. Now consider the set $\sigma_C(T)$ of those λ for which $\lambda I - T$ has a densely defined but unbounded inverse on its range,

$$\sigma_C(T) = \big\{\lambda \in \mathbb{C} \colon \mathcal{N}(\lambda I - T) = \{0\}, \ \mathcal{R}(\lambda I - T)^- = \mathcal{X} \ \text{and} \ \mathcal{R}(\lambda I - T) \neq \mathcal{X}\big\}$$

(see Theorem 1.3), which is referred to as the *continuous spectrum* of T. The *residual spectrum* of T is the set $\sigma_R(T)$ of all scalars λ such that $\lambda I - T$ has an inverse on its range that is not densely defined:

$$\sigma_R(T) = \big\{\lambda \in \mathbb{C} \colon \ \mathcal{N}(\lambda I - T) = \{0\} \ \text{and} \ \mathcal{R}(\lambda I - T)^- \neq \mathcal{X}\big\}.$$

The collection $\{\sigma_P(T), \sigma_C(T), \sigma_R(T)\}$ is a partition (i.e., it is a disjoint covering) of $\sigma(T)$, which means that they are pairwise disjoint and

$$\sigma(T) = \sigma_P(T) \cup \sigma_C(T) \cup \sigma_R(T).$$

The diagram below, borrowed from [62], summarizes such a partition of the spectrum. The residual spectrum is split into two disjoint parts, $\sigma_R(T) = \sigma_{R_1}(T) \cup \sigma_{R_2}(T)$, and the point spectrum into four disjoint parts, $\sigma_P(T) = \bigcup_{i=1}^{4} \sigma_{P_i}(T)$. We adopt the following abbreviated notation: $T_\lambda = \lambda I - T$, $\mathcal{N}_\lambda = \mathcal{N}(T_\lambda)$, and $\mathcal{R}_\lambda = \mathcal{R}(T_\lambda)$. Recall that if $\mathcal{N}(T_\lambda) = \{0\}$, then its linear inverse T_λ^{-1} on $\mathcal{R}_\lambda$ is continuous if and only if $\mathcal{R}_\lambda$ is closed (Theorem 1.2).

		$\mathcal{R}_\lambda^- = \mathcal{X}$		$\mathcal{R}_\lambda^- \neq \mathcal{X}$	
		$\mathcal{R}_\lambda^- = \mathcal{R}_\lambda$	$\mathcal{R}_\lambda^- \neq \mathcal{R}_\lambda$	$\mathcal{R}_\lambda^- \neq \mathcal{R}_\lambda$	$\mathcal{R}_\lambda^- = \mathcal{R}_\lambda$
$\mathcal{N}_\lambda = \{0\}$	$T_\lambda^{-1} \in \mathcal{B}[\mathcal{R}_\lambda, \mathcal{X}]$	$\rho(T)$	$\varnothing$	$\varnothing$	$\sigma_{R_1}(T)$
	$T_\lambda^{-1} \notin \mathcal{B}[\mathcal{R}_\lambda, \mathcal{X}]$	$\varnothing$	$\sigma_C(T)$	$\sigma_{R_2}(T)$	$\varnothing$
$\mathcal{N}_\lambda \neq \{0\}$		$\sigma_{P_1}(T)$	$\sigma_{P_2}(T)$	$\sigma_{P_3}(T)$	$\sigma_{P_4}(T)$

$$\left.\right\} \sigma_{AP}(T)$$

$$\underbrace{\qquad\qquad}_{\sigma_{CP}(T)}$$

Fig. § 2.2. A classical partition of the spectrum

Theorem 2.2 says that $\sigma(T) \neq \varnothing$, but any of the above disjoint parts of the spectrum may be empty (Section 2.7). However, if $\sigma_P(T) \neq \varnothing$, then a set of eigenvectors associated with *distinct* eigenvalues is linearly independent.

Theorem 2.3. *Let $\{\lambda_\gamma\}_{\gamma \in \Gamma}$ be a family of distinct eigenvalues of T. For each $\gamma \in \Gamma$ let x_γ be an eigenvector associated with λ_γ. The set $\{x_\gamma\}_{\gamma \in \Gamma}$ is linearly independent.*

Proof. For each $\gamma \in \Gamma$ take $0 \neq x_\gamma \in \mathcal{N}(\lambda_\gamma I - T) \neq \{0\}$, and consider the set $\{x_\gamma\}_{\gamma \in \Gamma}$ (whose existence is ensured by the Axiom of Choice).

Claim. Every finite subset of $\{x_\gamma\}_{\gamma \in \Gamma}$ is linearly independent.

Proof. Take an arbitrary finite subset of $\{x_\gamma\}_{\gamma \in \Gamma}$, say $\{x_i\}_{i=1}^{n}$. If $n = 1$, then linear independence is trivial. Suppose $\{x_i\}_{i=1}^{n}$ is linearly independent for some $n \geq 1$. If $\{x_i\}_{i=1}^{n+1}$ is not linearly independent, then $x_{n+1} = \sum_{i=1}^{n} \alpha_i x_i$, where the family $\{\alpha_i\}_{i=1}^{n}$ of complex numbers has at least one nonzero number. Thus

$$\lambda_{n+1} x_{n+1} = T x_{n+1} = \sum_{i=1}^{n} \alpha_i T x_i = \sum_{i=1}^{n} \alpha_i \lambda_i x_i.$$

If $\lambda_{n+1} = 0$, then $\lambda_i \neq 0$ for every $i \neq n+1$ (because the eigenvalues are distinct) and $\sum_{i=1}^{n} \alpha_i \lambda_i x_i = 0$ so that $\{x_i\}_{i=1}^{n}$ is not linearly independent, which is a contradiction. If $\lambda_{n+1} \neq 0$, then $x_{n+1} = \sum_{i=1}^{n} \alpha_i \lambda_{n+1}^{-1} \lambda_i x_i$, and therefore $\sum_{i=1}^{n} \alpha_i (1 - \lambda_{n+1}^{-1} \lambda_i) x_i = 0$ so that $\{x_i\}_{i=1}^{n}$ is not linearly independent (since $\lambda_i \neq \lambda_{n+1}$ for every $i \neq n+1$ and $\alpha_i \neq 0$ for some i), which is again a contradiction. This completes the proof by induction: $\{x_i\}_{i=1}^{n+1}$ is linearly independent.

However, if every finite subset of $\{x_\gamma\}_{\gamma \in \Gamma}$ is linearly independent, then so is the set $\{x_\gamma\}_{\gamma \in \Gamma}$ itself (see, e.g., [66, Proposition 2.3]). $\square$

There are some overlapping parts of the spectrum which are commonly used. For instance, the *compression spectrum* $\sigma_{CP}(T)$ and the *approximate point spectrum* (or *approximation spectrum*) $\sigma_{AP}(T)$, which are defined by

$$\sigma_{CP}(T) = \big\{\lambda \in \mathbb{C}\colon\ \mathcal{R}(\lambda I - T) \text{ is not dense in } \mathcal{X}\big\}$$
$$= \sigma_{P_3}(T) \cup \sigma_{P_4}(T) \cup \sigma_R(T),$$

$$\sigma_{AP}(T) = \big\{\lambda \in \mathbb{C}\colon\ \lambda I - T \text{ is not bounded below}\big\}$$
$$= \sigma_P(T) \cup \sigma_C(T) \cup \sigma_{R_2}(T) \ = \ \sigma(T)\backslash\sigma_{R_1}(T).$$

The points of $\sigma_{AP}(T)$ are referred to as the *approximate eigenvalues* of T.

Theorem 2.4. *The following assertions are pairwise equivalent.*

(a) *For every $\varepsilon > 0$ there is a unit vector x_ε in $\mathcal{X}$ such that $\|(\lambda I - T)x_\varepsilon\| < \varepsilon$.*

(b) *There is a sequence $\{x_n\}$ of unit vectors in $\mathcal{X}$ such that $\|(\lambda I - T)x_n\| \to 0$.*

(c) $\lambda \in \sigma_{AP}(T)$.

Proof. Clearly (a) implies (b). If (b) holds, then there is no constant $\alpha > 0$ such that $\alpha = \alpha\|x_n\| \le \|(\lambda I - T)x_n\|$ for all n. Thus $\lambda I - T$ is not bounded below, and so (b) implies (c). Conversely, if $\lambda I - T$ is not bounded below, then there is no constant $\alpha > 0$ such that $\alpha\|x\| \le \|(\lambda I - T)x\|$ for all $x \in \mathcal{X}$ or, equivalently, for every $\varepsilon > 0$ there exists a nonzero y_ε in $\mathcal{X}$ such that $\|(\lambda I - T)y_\varepsilon\| < \varepsilon\|y_\varepsilon\|$. Set $x_\varepsilon = \|y_\varepsilon\|^{-1}y_\varepsilon$, and hence (c) implies (a). $\square$

Theorem 2.5. *The approximate point spectrum $\sigma_{AP}(T)$ is a nonempty closed subset of $\mathbb{C}$ that includes the boundary $\partial\sigma(T)$ of the spectrum $\sigma(T)$.*

Proof. Take an arbitrary λ in $\partial\sigma(T)$. Recall that $\rho(T) \ne \varnothing$, $\sigma(T)$ is closed (Theorem 2.1), and $\partial\sigma(T) = \partial\rho(T) = \rho(T)^- \cap \sigma(T)$. Thus λ is a point of adherence of $\rho(T)$, and so there exists a sequence $\{\lambda_n\}$ with each λ_n in $\rho(T)$ such that $\lambda_n \to \lambda$. Since

$$(\lambda_n I - T) - (\lambda I - T) = (\lambda_n - \lambda)I$$

for every n, it follows that $\lambda_n I - T \to \lambda I - T$ in $\mathcal{B}[\mathcal{X}]$.

Claim. $\sup_n \|(\lambda_n I - T)^{-1}\| = \infty$.

Proof. Since each λ_n lies in $\rho(T)$ and $\lambda \in \partial\sigma(T)$ does not lie in $\rho(T)$ (because $\sigma(T)$ is closed), it follows that the sequence $\{\lambda_n I - T\}$ of operators in $\mathcal{G}[\mathcal{X}]$ converges in $\mathcal{B}[\mathcal{X}]$ to $\lambda I - T \in \mathcal{B}[\mathcal{X}]\backslash\mathcal{G}[\mathcal{X}]$. If $\lambda \in \sigma_P(T)$, then there exists $x \ne 0$ in $\mathcal{X}$ such that, if $\sup_n \|(\lambda_n I - T)^{-1}\| < \infty$,

$$\begin{aligned}
0 \ne \|x\| &= \|(\lambda_n I - T)^{-1}(\lambda_n I - T)x\| \\
&\le \sup_n \|(\lambda_n I - T)^{-1}\| \limsup_n \|(\lambda_n I - T)x\| \\
&\le \sup_n \|(\lambda_n I - T)^{-1}\| \, \|(\lambda I - T)x\| = 0,
\end{aligned}$$

which is a contradiction. If $\lambda \notin \sigma_P(T)$, then $\mathcal{N}(\lambda I - T) = \{0\}$, and hence there exists the inverse $(\lambda I - T)^{-1}$ on $\mathcal{R}(\lambda I - T)$ so that, for each n,

$$(\lambda_n I - T)^{-1} - (\lambda I - T)^{-1} = (\lambda_n I - T)^{-1}[(\lambda I - T) - (\lambda_n I - T)](\lambda I - T)^{-1}.$$

If $\sup_n \|(\lambda_n I - T)^{-1}\| < \infty$, then $(\lambda_n I - T)^{-1} \to (\lambda I - T)^{-1}$ in $\mathcal{B}[\mathcal{X}]$, because $(\lambda_n I - T) \to (\lambda I - T)$ in $\mathcal{B}[\mathcal{X}]$, and so $(\lambda I - T)^{-1} \in \mathcal{B}[\mathcal{X}]$. That is, $(\lambda I - T) \in \mathcal{G}[\mathcal{X}]$, which is again a contradiction (since $\lambda \in \sigma(T)$). This completes the proof of the claimed result.

Since $\|(\lambda_n I - T)^{-1}\| = \sup_{\|y\|=1} \|(\lambda_n I - T)^{-1}y\|$, take a unit vector y_n in $\mathcal{X}$ for which $\|(\lambda_n I - T)^{-1}\| - \frac{1}{n} \leq \|(\lambda_n I - T)^{-1}y_n\| \leq \|(\lambda_n I - T)^{-1}\|$ for each n. The above claim ensures that $\sup_n \|(\lambda_n I - T)^{-1}y_n\| = \infty$, and hence $\inf_n \|(\lambda_n I - T)^{-1}y_n\|^{-1} = 0$, so that there exist subsequences of $\{\lambda_n\}$ and $\{y_n\}$, say $\{\lambda_k\}$ and $\{y_k\}$, for which

$$\|(\lambda_k I - T)^{-1}y_k\|^{-1} \to 0.$$

Set $x_k = \|(\lambda_k I - T)^{-1}y_k\|^{-1}(\lambda_k I - T)^{-1}y_k$ and get a sequence $\{x_k\}$ of unit vectors in $\mathcal{X}$ such that $\|(\lambda_k I - T)x_k\| = \|(\lambda_k I - T)^{-1}y_k\|^{-1}$. Hence

$$\|(\lambda I - T)x_k\| = \|(\lambda_k I - T)x_k - (\lambda_k - \lambda)x_k\| \leq \|(\lambda_k I - T)^{-1}y_k\|^{-1} + |\lambda_k - \lambda|.$$

Since $\lambda_k \to \lambda$, it then follows that $\|(\lambda I - T)x_k\| \to 0$, and so $\lambda \in \sigma_{AP}(T)$ according to Theorem 2.4. Therefore,

$$\partial\sigma(T) \subseteq \sigma_{AP}(T).$$

This inclusion clearly implies that $\sigma_{AP}(T) \neq \varnothing$ (for $\sigma(T)$ is closed and nonempty). Finally, take an arbitrary $\lambda \in \mathbb{C}\backslash\sigma_{AP}(T)$ so that $\lambda I - T$ is bounded below. Therefore, there exists an $\alpha > 0$ for which

$$\alpha\|x\| \leq \|(\lambda I - T)x\| \leq \|(\nu I - T)x\| + \|(\lambda - \nu)x\|,$$

and hence $(\alpha - |\lambda - \nu|)\|x\| \leq \|(\nu I - T)x\|$, for all $x \in \mathcal{X}$ and $\nu \in \mathbb{C}$. Then $\nu I - T$ is bounded below for every ν such that $0 < \alpha - |\lambda - \nu|$. Equivalently, $\nu \in \mathbb{C}\backslash\sigma_{AP}(T))$ for every ν sufficiently close to λ (i.e., if $|\lambda - \nu| < \alpha$). Thus the nonempty open ball $B_\alpha(\lambda)$ centered at λ is included in $\mathbb{C}\backslash\sigma_{AP}(T)$. Therefore $\mathbb{C}\backslash\sigma_{AP}(T)$ is open, and so $\sigma_{AP}(T)$ is closed. $\qquad\square$

Remark. $\sigma_{AP}(T) = \sigma(T)\backslash\sigma_{R_1}(T)$ is closed in $\mathbb{C}$ and includes $\partial\sigma(T) = \partial\rho(T)$. So $\mathbb{C}\backslash\sigma_{R_1}(T) = \rho(T) \cup \sigma_{AP}(T) = \rho(T) \cup \partial\rho(T) \cup \sigma_{AP}(T) = \rho(T)^- \cup \sigma_{AP}(T)$ is closed in $\mathbb{C}$. Outcome:

$$\sigma_{R_1}(T) \text{ is an open subset of } \mathbb{C}.$$

For the remainder of this section we assume that T lies in $\mathcal{B}[\mathcal{H}]$, where $\mathcal{H}$ is a *nonzero complex Hilbert space*. This will bring forth some important

simplifications. A particularly useful instance of such simplifications is the formula for the residual spectrum in the next theorem. First we need the following piece of notation. If Λ is any subset of $\mathbb{C}$, then set

$$\Lambda^* = \left\{ \overline{\lambda} \in \mathbb{C} : \lambda \in \Lambda \right\}$$

so that $\Lambda^{**} = \Lambda$, $(\mathbb{C}\backslash\Lambda)^* = \mathbb{C}\backslash\Lambda^*$, and $(\Lambda_1 \cup \Lambda_2)^* = \Lambda_1^* \cup \Lambda_2^*$.

Theorem 2.6. *If* $T^* \in \mathcal{B}[\mathcal{H}]$ *is the adjoint of* $T \in \mathcal{B}[\mathcal{H}]$, *then*

$$\rho(T) = \rho(T^*)^*, \qquad \sigma(T) = \sigma(T^*)^*, \qquad \sigma_C(T) = \sigma_C(T^*)^*,$$

and the residual spectrum of T *is given by the formula*

$$\sigma_R(T) = \sigma_P(T^*)^*\backslash\sigma_P(T).$$

As for the subparts of the point and residual spectra,

$$\sigma_{P_1}(T) = \sigma_{R_1}(T^*)^*, \qquad \sigma_{P_2}(T) = \sigma_{R_2}(T^*)^*,$$

$$\sigma_{P_3}(T) = \sigma_{P_3}(T^*)^*, \qquad \sigma_{P_4}(T) = \sigma_{P_4}(T^*)^*.$$

For the compression and approximate point spectra we get

$$\sigma_{CP}(T) = \sigma_P(T^*)^*, \qquad \sigma_{AP}(T^*)^* = \sigma(T)\backslash\sigma_{P_1}(T),$$

$$\partial\sigma(T) \subseteq \sigma_{AP}(T) \cap \sigma_{AP}(T^*)^* = \sigma(T)\backslash(\sigma_{P_1}(T) \cup \sigma_{R_1}(T)).$$

Proof. Since $S \in \mathcal{G}[\mathcal{H}]$ if and only if $S^* \in \mathcal{G}[\mathcal{H}]$, we get $\rho(T) = \rho(T^*)^*$. Hence $\sigma(T)^* = (\mathbb{C}\backslash\rho(T))^* = \mathbb{C}\backslash\rho(T^*) = \sigma(T^*)$. Recall that $\mathcal{R}(S)^- = \mathcal{R}(S)$ if and only if $\mathcal{R}(S^*)^- = \mathcal{R}(S^*)$, and $\mathcal{N}(S) = \{0\}$ if and only if $\mathcal{R}(S^*)^\perp = \{0\}$ (cf. Lemmas 1.4 and 1.5), which means that $\mathcal{R}(S^*)^- = \mathcal{H}$. Thus $\sigma_{P_1}(T) = \sigma_{R_1}(T^*)^*$, $\sigma_{P_2}(T) = \sigma_{R_2}(T^*)^*$, $\sigma_{P_3}(T) = \sigma_{P_3}(T^*)^*$, and $\sigma_{P_4}(T) = \sigma_{P_4}(T^*)^*$. Applying the same argument, $\sigma_C(T) = \sigma_C(T^*)^*$ and $\sigma_{CP}(T) = \sigma_P(T^*)^*$. Hence,

$$\sigma_R(T) = \sigma_{CP}(T)\backslash\sigma_P(T) \quad \text{implies} \quad \sigma_R(T) = \sigma_P(T^*)^*\backslash\sigma_P(T).$$

Moreover, by using the above properties and the definition of $\sigma_{AP}(T^*) = \sigma_P(T^*) \cup \sigma_C(T^*) \cup \sigma_{R_2}(T^*) = \sigma_{CP}(T)^* \cup \sigma_C(T)^* \cup \sigma_{P_2}(T)^*$, we get

$$\sigma_{AP}(T^*)^* = \sigma_{CP}(T) \cup \sigma_C(T) \cup \sigma_{P_2}(T) = \sigma(T)\backslash\sigma_{P_1}(T).$$

Therefore, $\sigma_{AP}(T^*)^* \cap \sigma_{AP}(T) = \sigma(T)\backslash(\sigma_{P_1}(T) \cup \sigma_{R_1}(T))$. But $\sigma(T)$ is closed and $\sigma_{R_1}(T)$ is open (and so is $\sigma_{P_1}(T) = \sigma_{R_1}(T^*)^*$) in $\mathbb{C}$. This implies that $\sigma_{P_1}(T) \cup \sigma_{R_1}(T) \subseteq \sigma(T)^\circ$, where $\sigma(T)^\circ$ denotes the interior of $\sigma(T)$, and $\partial\sigma(T) \subseteq \sigma(T)\backslash(\sigma_{P_1}(T) \cup \sigma_{R_1}(T))$. $\qquad\square$

Remark. We have just shown that

$$\sigma_{P_1}(T) \text{ is an open subset of } \mathbb{C}.$$

2.3 Spectral Mapping

Spectral Mapping Theorems are pivotal results in spectral theory. Here we focus on the important particular case for polynomials. Further versions will be considered in Chapter 4. Let $p \colon \mathbb{C} \to \mathbb{C}$ be a polynomial with complex coefficients, take any subset Λ of $\mathbb{C}$, and consider its image under p, viz.,

$$p(\Lambda) = \{p(\lambda) \in \mathbb{C} \colon \ \lambda \in \Lambda\}.$$

Theorem 2.7. (Spectral Mapping Theorem for Polynomials). *Take an operator $T \in \mathcal{B}[\mathcal{X}]$ on a complex Banach space $\mathcal{X}$. If p is any polynomial with complex coefficients, then*

$$\sigma(p(T)) = p(\sigma(T)).$$

Proof. To avoid trivialities, let $p \colon \mathbb{C} \to \mathbb{C}$ be an arbitrary *nonconstant* polynomial with complex coefficients,

$$p(\lambda) = \sum_{i=0}^{n} \alpha_i \lambda^i, \quad \text{with } n \geq 1 \text{ and } \alpha_n \neq 0,$$

for every $\lambda \in \mathbb{C}$. Take an arbitrary $\nu \in \mathbb{C}$ and consider the factorization

$$\nu - p(\lambda) = \beta_n \prod_{i=1}^{n} (\lambda_i - \lambda),$$

with $\beta_n = -1^{n+1} \alpha_n$ where $\{\lambda_i\}_{i=1}^{n}$ are the roots of $\nu - p(\lambda)$, so that

$$\nu I - p(T) = \nu I - \sum_{i=0}^{n} \alpha_i T^i = \beta_n \prod_{i=1}^{n} (\lambda_i I - T).$$

If $\lambda_i \in \rho(T)$ for every $i = 1, \ldots, n$, then $\beta_n \prod_{i=1}^{n} (\lambda_i I - T) \in \mathcal{G}[\mathcal{X}]$ so that $\nu \in \rho(p(T))$. Thus if $\nu \in \sigma(p(T))$, then there exists $\lambda_j \in \sigma(T)$ for some $j = 1, \ldots, n$. However, λ_j is a root of $\nu - p(\lambda)$, that is,

$$\nu - p(\lambda_j) = \beta_n \prod_{i=1}^{n} (\lambda_i - \lambda_j) = 0,$$

and so $p(\lambda_j) = \nu$. Hence, if $\nu \in \sigma(p(T))$, then

$$\nu = p(\lambda_j) \in \{p(\lambda) \in \mathbb{C} \colon \lambda \in \sigma(T)\} = p(\sigma(T))$$

because $\lambda_j \in \sigma(T)$, and therefore

$$\sigma(p(T)) \subseteq p(\sigma(T)).$$

Conversely, if $\nu \in p(\sigma(T)) = \{p(\lambda) \in \mathbb{C} \colon \lambda \in \sigma(T)\}$, then $\nu = p(\lambda)$ for some $\lambda \in \sigma(T)$. Thus $\nu - p(\lambda) = 0$ so that $\lambda = \lambda_j$ for some $j = 1, \ldots, n$, and hence

$$\nu I - p(T) = \beta_n \prod_{i=1}^{n} (\lambda_i I - T)$$

$$= (\lambda_j I - T)\beta_n \prod_{j\neq i=1}^{n} (\lambda_i I - T) = \beta_n \prod_{j\neq i=1}^{n} (\lambda_i I - T)(\lambda_j I - T)$$

since $(\lambda_j I - T)$ commutes with $(\lambda_i I - T)$ for every i. If $\nu \in \rho(p(T))$, then $(\nu I - p(T)) \in \mathcal{G}[\mathcal{X}]$ so that

$$(\lambda_j I - T)\left(\beta_n \prod_{j\neq i=1}^{n} (\lambda_i I - T)(\nu I - p(T))^{-1}\right)$$

$$= (\nu I - p(T))(\nu I - p(T))^{-1} = I = (\nu I - p(T))^{-1}(\nu I - p(T))$$

$$= \left((\nu I - p(T))^{-1}\beta_n \prod_{j\neq i=1}^{n} (\lambda_i I - T)\right)(\lambda_j I - T).$$

Then $(\lambda_j I - T)$ has a right and a left inverse (i.e., it is invertible), and so $(\lambda_j I - T) \in \mathcal{G}[\mathcal{X}]$ by Theorem 1.1. Hence, $\lambda = \lambda_j \in \rho(T)$, which contradicts the fact that $\lambda \in \sigma(T)$. Conclusion: if $\nu \in p(\sigma(T))$, then $\nu \notin \rho(p(T))$, that is, $\nu \in \sigma(p(T))$. Thus

$$p(\sigma(T)) \subseteq \sigma(p(T)). \qquad \square$$

In particular,
$$\sigma(T^n) = \sigma(T)^n \quad \text{for every } n \geq 0,$$
that is, $\nu \in \sigma(T)^n = \{\lambda^n \in \mathbb{C} \colon \lambda \in \sigma(T)\}$ if and only if $\nu \in \sigma(T^n)$, and

$$\sigma(\alpha T) = \alpha\sigma(T) \quad \text{for every } \alpha \in \mathbb{C},$$

that is, $\nu \in \alpha\sigma(T) = \{\alpha\lambda \in \mathbb{C} \colon \lambda \in \sigma(T)\}$ if and only if $\nu \in \sigma(\alpha T)$. Also notice (even though this is not a particular case of the Spectral Mapping Theorem for polynomials) that if $T \in \mathcal{G}[\mathcal{X}]$, then

$$\sigma(T^{-1}) = \sigma(T)^{-1},$$

which means that $\nu \in \sigma(T)^{-1} = \{\lambda^{-1} \in \mathbb{C} \colon 0 \neq \lambda \in \sigma(T)\}$ if and only if $\nu \in \sigma(T^{-1})$. Indeed, if $T \in \mathcal{G}[\mathcal{X}]$ (so that $0 \in \rho(T)$) and if $\nu \neq 0$, then $-\nu T^{-1}(\nu^{-1}I - T) = \nu I - T^{-1}$. Thus $\nu^{-1} \in \rho(T)$ if and only if $\nu \in \rho(T^{-1})$. Also note that if $T \in \mathcal{B}[\mathcal{H}]$, then

$$\sigma(T^*) = \sigma(T)^*$$

by Theorem 2.6, where $\mathcal{H}$ is a complex Hilbert space.

The next result is an extension of the Spectral Mapping Theorem for polynomials which holds for normal operators in a Hilbert space $\mathcal{H}$. If Λ_1 and Λ_2 are arbitrary subsets of $\mathbb{C}$ and $p \colon \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is any polynomial in two variables (with complex coefficients), then set

$$p(\Lambda_1, \Lambda_2) = \big\{p(\lambda_1, \lambda_2) \in \mathbb{C}: \ \lambda_1 \in \Lambda_1, \ \lambda_2 \in \Lambda_2\big\};$$

in particular, with $\Lambda^* = \{\overline{\lambda} \in \mathbb{C}: \ \lambda \in \Lambda\}$,

$$p(\Lambda, \Lambda^*) = \big\{p(\lambda, \overline{\lambda}) \in \mathbb{C}: \ \lambda \in \Lambda\big\}.$$

Theorem 2.8. (Spectral Mapping Theorem for Normal Operators). *If $T \in \mathcal{B}[\mathcal{H}]$ is normal and $p(\cdot,\cdot)$ is a polynomial in two variables, then*

$$\sigma(p(T,T^*)) = p(\sigma(T), \sigma(T^*)) = \big\{p(\lambda,\overline{\lambda}) \in \mathbb{C}: \ \lambda \in \sigma(T)\big\}.$$

Proof. Take any normal operator $T \in \mathcal{B}[\mathcal{H}]$. If $p(\lambda, \overline{\lambda}) = \sum_{i,j=0}^{n,m} \alpha_{i,j}\lambda^i\overline{\lambda}^j$, then set $p(T,T^*) = \sum_{i,j=0}^{n,m} \alpha_{i,j}T^iT^{*j} = p(T^*,T)$. Let $\mathcal{P}(T,T^*)$ be the collection of all those polynomials $p(T,T^*)$, which is a commutative subalgebra of $\mathcal{B}[\mathcal{H}]$ since T commutes with T^*. Consider the collection $\mathcal{T}$ of all commutative subalgebras of $\mathcal{B}[\mathcal{H}]$ containing T and T^*, which is partially ordered (in the inclusion ordering) and nonempty (e.g., $\mathcal{P}(T,T^*) \in \mathcal{T}$). Moreover, every chain in $\mathcal{T}$ has an upper bound in $\mathcal{T}$ (the union of all subalgebras in a given chain of subalgebras in $\mathcal{T}$ is again a subalgebra in $\mathcal{T}$). Thus Zorn's Lemma says that $\mathcal{T}$ has a maximal element, say $\mathcal{A}(T)$. Outcome: If T is normal, then there is a maximal (thus closed) commutative subalgebra $\mathcal{A}(T)$ of $\mathcal{B}[\mathcal{H}]$ containing T and T^*. Since $\mathcal{P}(T,T^*) \subseteq \mathcal{A}(T) \in \mathcal{T}$, and every $p(T,T^*) \in \mathcal{P}(T,T^*)$ is normal, $\mathcal{A}(p(T,T^*)) = \mathcal{A}(T)$ for every nonconstant $p(T,T^*)$. Furthermore,

$$\Phi(p(T,T^*)) = p(\Phi(T), \Phi(T^*))$$

for every homomorphism $\Phi\colon \mathcal{A}(T) \to \mathbb{C}$. Thus, by Proposition 2.Q(b),

$$\sigma(p(T,T^*)) = \big\{p(\Phi(T), \Phi(T^*)) \in \mathbb{C}: \ \Phi \in \widehat{\mathcal{A}}(T)\big\}.$$

Take a surjective homomorphism $\Phi\colon \mathcal{A}(T) \to \mathbb{C}$ (i.e., take any $\Phi \in \widehat{\mathcal{A}}(T)$). Consider the Cartesian decomposition $T = A + iB$, where $A, B \in \mathcal{B}[\mathcal{H}]$ are self-adjoint, and so $T^* = A - iB$ (Proposition 1.O). Thus $\Phi(T) = \Phi(A) + i\Phi(B)$ and $\Phi(T^*) = \Phi(A) - i\Phi(B)$. Since $A = \frac{1}{2}(T + T^*)$ and $B = -\frac{i}{2}(T - T^*)$ lie in $\mathcal{P}(T,T^*)$, we get $\mathcal{A}(A) = \mathcal{A}(B) = \mathcal{A}(T)$. Moreover, since they are self-adjoint, $\{\Phi(A) \in \mathbb{C}: \ \Phi \in \widehat{\mathcal{A}}(T)\} = \sigma(A) \subset \mathbb{R}$ and $\{\Phi(B) \in \mathbb{C}: \ \Phi \in \widehat{\mathcal{A}}(T)\} = \sigma(B) \subset \mathbb{R}$ (Propositions 2.A and 2.Q(b)), and so $\Phi(A) \in \mathbb{R}$ and $\Phi(B) \in \mathbb{R}$. Hence

$$\Phi(T^*) = \overline{\Phi(T)}.$$

Therefore, since $\sigma(T^*) = \sigma(T)^*$ for every $T \in \mathcal{B}[\mathcal{H}]$ by Theorem 2.6, and according to Proposition 2.Q(b),

$$
\begin{aligned}
\sigma(p(T,T^*)) &= \big\{p(\Phi(T), \overline{\Phi(T)}) \in \mathbb{C}: \ \Phi \in \widehat{\mathcal{A}}(T)\big\} \\
&= \big\{p(\lambda, \overline{\lambda}) \in \mathbb{C}: \ \lambda \in \{\Phi(T) \in \mathbb{C}: \ \Phi \in \widehat{\mathcal{A}}(T)\}\big\} \\
&= \big\{p(\lambda, \overline{\lambda}) \in \mathbb{C}: \ \lambda \in \sigma(T)\big\} \\
&= p(\sigma(T), \sigma(T)^*) = p(\sigma(T), \sigma(T^*)). \qquad \square
\end{aligned}
$$

2.4 Spectral Radius

The *spectral radius* of an operator $T \in \mathcal{B}[\mathcal{X}]$ on a nonzero complex Banach space $\mathcal{X}$ is the nonnegative number

$$r_\sigma(T) \;=\; \sup_{\lambda \in \sigma(T)} |\lambda| \;=\; \max_{\lambda \in \sigma(T)} |\lambda|.$$

The first identity in the above expression defines the spectral radius $r_\sigma(T)$, and the second one is a consequence of the Weierstrass Theorem (cf. proof of Theorem 2.2) since $\sigma(T) \neq \varnothing$ is compact in $\mathbb{C}$ and the function $|\;|\colon \mathbb{C} \to \mathbb{R}$ is continuous. A straightforward consequence of the Spectral Mapping Theorem for polynomials reads as follows.

Corollary 2.9. $r_\sigma(T^n) = r_\sigma(T)^n$ *for every* $n \geq 0$.

Proof. Take an arbitrary nonnegative integer n. Theorem 2.7 ensures that $\sigma(T^n) = \sigma(T)^n$. Hence $\nu \in \sigma(T^n)$ if and only if $\nu = \lambda^n$ for some $\lambda \in \sigma(T)$, and so $\sup_{\nu \in \sigma(T^n)} |\nu| = \sup_{\lambda \in \sigma(T)} |\lambda^n| = \sup_{\lambda \in \sigma(T)} |\lambda|^n = \left(\sup_{\lambda \in \sigma(T)} |\lambda|\right)^n$. $\square$

If $\lambda \in \sigma(T)$, then $|\lambda| \leq \|T\|$. This follows by the Neumann expansion of Theorem 1.3 (cf. proof of Theorem 2.1). Thus $r_\sigma(T) \leq \|T\|$. Therefore, for every operator $T \in \mathcal{B}[\mathcal{X}]$, and for each nonnegative integer n,

$$0 \leq r_\sigma(T^n) = r_\sigma(T)^n \leq \|T^n\| \leq \|T\|^n.$$

Thus $r_\sigma(T) \leq 1$ if T is *power bounded* (i.e., if $\sup_n \|T^n\| < \infty$). Indeed, in this case, $r_\sigma(T)^n = r_\sigma(T^n) \leq \sup_k \|T^k\|$ and $\lim_n (\sup_k \|T^k\|)^{\frac{1}{n}} = 1$, and so

$$\sup_n \|T^n\| < \infty \quad \text{implies} \quad r_\sigma(T) \leq 1.$$

Remark. If T is a *nilpotent* operator (i.e., if $T^n = O$ for some $n \geq 1$), then $r_\sigma(T) = 0$, and so $\sigma(T) = \sigma_P(T) = \{0\}$ (cf. Proposition 2.J). An operator $T \in \mathcal{B}[\mathcal{X}]$ is *quasinilpotent* if $r_\sigma(T) = 0$ (i.e., if $\sigma(T) = \{0\}$). Thus every nilpotent is quasinilpotent. Since $\sigma_P(T)$ may be empty for a quasinilpotent operator (cf. Proposition 2.N), these classes are related by proper inclusion:

$$\text{Nilpotent} \quad \subset \quad \text{Quasinilpotent}.$$

The next result is the well-known Gelfand–Beurling formula for the spectral radius. Its proof requires another piece of elementary complex analysis, viz., *every analytic function has a power series representation*. That is, if $f \colon \Lambda \to \mathbb{C}$ is analytic, and if $B_{\alpha,\beta}(\nu) = \{\lambda \in \mathbb{C} \colon 0 \leq \alpha < |\lambda - \nu| < \beta\}$ lies in the open set $\Lambda \subseteq \mathbb{C}$, then f has a *unique Laurent expansion* about the point ν, namely, $f(\lambda) = \sum_{k=-\infty}^{\infty} \gamma_k (\lambda - \nu)^k$ for every $\lambda \in B_{\alpha,\beta}(\nu)$.

Theorem 2.10. (GELFAND–BEURLING FORMULA).

$$r_\sigma(T) = \lim_n \|T^n\|^{\frac{1}{n}}.$$

Proof. Since $r_\sigma(T)^n \leq \|T^n\|$ for every positive integer n, and since the limit of the sequence $\{\|T^n\|^{\frac{1}{n}}\}$ exists by Lemma 1.10, we get

$$r_\sigma(T) \leq \lim_n \|T^n\|^{\frac{1}{n}}.$$

For the reverse inequality, proceed as follows. Consider the Neumann expansion (Theorem 1.3) for the resolvent function $R_T \colon \rho(T) \to \mathcal{G}[\mathcal{X}]$,

$$R_T(\lambda) = (\lambda I - T)^{-1} = \lambda^{-1} \sum_{k=0}^{\infty} T^k \lambda^{-k}$$

for every $\lambda \in \rho(T)$ such that $\|T\| < |\lambda|$, where the above series converges in the (uniform) topology of $\mathcal{B}[\mathcal{X}]$. Take an arbitrary bounded linear functional $\eta \colon \mathcal{B}[\mathcal{X}] \to \mathbb{C}$ in $\mathcal{B}[\mathcal{X}]^*$ (cf. proof of Theorem 2.2). Since η is continuous,

$$\eta(R_T(\lambda)) = \lambda^{-1} \sum_{k=0}^{\infty} \eta(T^k) \lambda^{-k}$$

for every $\lambda \in \rho(T)$ such that $\|T\| < |\lambda|$.

Claim. The above displayed identity holds whenever $r_\sigma(T) < |\lambda|$.

Proof. $\lambda^{-1} \sum_{k=0}^{\infty} \eta(T^k) \lambda^{-k}$ is a Laurent expansion of $\eta(R_T(\lambda))$ about the origin for every $\lambda \in \rho(T)$ such that $\|T\| < |\lambda|$. But $\eta \circ R_T$ is analytic on $\rho(T)$ (cf. Claim 2 in Theorem 2.2) so that $\eta(R_T(\lambda))$ has a unique Laurent expansion about the origin for every $\lambda \in \rho(T)$, and hence for every $\lambda \in \mathbb{C}$ such that $r_\sigma(T) < |\lambda|$. Then $\eta(R_T(\lambda)) = \lambda^{-1} \sum_{k=0}^{\infty} \eta(T^k) \lambda^{-k}$, which holds whenever $r_\sigma(T) \leq \|T\| < |\lambda|$, must be the Laurent expansion about the origin for every $\lambda \in \mathbb{C}$ such that $r_\sigma(T) < |\lambda|$, thus proving the claimed result.

Hence, if $r_\sigma(T) < |\lambda|$, then the series of complex numbers $\sum_{k=0}^{\infty} \eta(T^k) \lambda^{-k}$ converges, and so $\eta((\lambda^{-1}T)^k) = \eta(T^k)\lambda^{-k} \to 0$, for every η in the dual space $\mathcal{B}[\mathcal{X}]^*$ of $\mathcal{B}[\mathcal{X}]$. This means that the $\mathcal{B}[\mathcal{X}]$-valued sequence $\{(\lambda^{-1}T)^k\}$ converges weakly. Then it is bounded (in the uniform topology of $\mathcal{B}[\mathcal{X}]$ as a consequence of the Banach–Steinhaus Theorem). That is, the operator $\lambda^{-1}T$ is power bounded. Thus $|\lambda|^{-n}\|T^n\| \leq \sup_k \|(\lambda^{-1}T)^k\| < \infty$, so that

$$|\lambda|^{-1}\|T^n\|^{\frac{1}{n}} \leq \big(\sup_k \|(\lambda^{-1}T)^k\|\big)^{\frac{1}{n}},$$

for every n. Therefore, $|\lambda|^{-1} \lim_n \|T^n\|^{\frac{1}{n}} \leq 1$, and so $\lim_n \|T^n\|^{\frac{1}{n}} \leq |\lambda|$ for every $\lambda \in \mathbb{C}$ such that $r_\sigma(T) < |\lambda|$. That is, $\lim_n \|T^n\|^{\frac{1}{n}} \leq r_\sigma(T) + \varepsilon$ for every $\varepsilon > 0$. Outcome:

$$\lim_n \|T^n\|^{\frac{1}{n}} \leq r_\sigma(T). \qquad \square$$

What Theorem 2.10 says is that $r_\sigma(T) = r(T)$, where $r_\sigma(T)$ is the spectral radius of T and $r(T)$ is the limit of the sequence $\{\|T^n\|^{\frac{1}{n}}\}$ (whose existence was proved in Lemma 1.10). We shall then adopt one and the same notation (the simplest, of course) for both of them. Thus, from now on, we write

$$r(T) = \sup_{\lambda \in \sigma(T)} |\lambda| = \max_{\lambda \in \sigma(T)} |\lambda| = \lim_{n} \|T^n\|^{\frac{1}{n}}.$$

A normaloid was defined in Section 1.6 as an operator T for which $r(T) = \|T\|$. Since $r_\sigma(T) = r(T)$, it follows that *a normaloid operator acting on a complex Banach space is precisely an operator whose norm coincides with the spectral radius*. Moreover, since on a complex Hilbert space $\mathcal{H}$ every normal operator is normaloid, and so is every nonnegative operator, and since T^*T is always nonnegative, it follows that, for every $T \in \mathcal{B}[\mathcal{H}]$,

$$r(T^*T) = r(TT^*) = \|T^*T\| = \|TT^*\| = \|T\|^2 = \|T^*\|^2.$$

Further useful properties of the spectral radius follow from Theorem 2.10:

$$r(\alpha T) = |\alpha|\, r(T) \quad \text{for every } \alpha \in \mathbb{C}$$

and, if $\mathcal{H}$ is a complex Hilbert space and $T \in \mathcal{B}[\mathcal{H}]$, then

$$r(T^*) = r(T).$$

An important application of the Gelfand–Beurling formula is the characterization of *uniform stability* in terms of the spectral radius. An operator T in $\mathcal{B}[\mathcal{X}]$ is *uniformly stable* if the power sequence $\{T^n\}$ converges uniformly to the null operator (i.e., if $\|T^n\| \to 0$). Notation: $T^n \xrightarrow{u} O$.

Corollary 2.11. *If $T \in \mathcal{B}[\mathcal{X}]$ is an operator on a complex Banach space $\mathcal{X}$, then the following assertions are pairwise equivalent.*

(a) $T^n \xrightarrow{u} O$.

(b) $r(T) < 1$.

(c) $\|T^n\| \leq \beta\alpha^n$ *for every $n \geq 0$, for some $\beta \geq 1$ and some $\alpha \in (0,1)$.*

Proof. Since $r(T)^n = r(T^n) \leq \|T^n\|$ for each $n \geq 1$, it follows that $\|T^n\| \to 0$ implies $r(T) < 1$. Now suppose $r(T) < 1$ and take any α in $(r(T), 1)$. Since $r(T) = \lim_n \|T^n\|^{\frac{1}{n}}$ (Gelfand–Beurling formula), there is an integer $n_\alpha \geq 1$ such that $\|T^n\| \leq \alpha^n$ for every $n \geq n_\alpha$. Thus $\|T^n\| \leq \beta\alpha^n$ for every $n \geq 0$ with $\beta = \max_{0 \leq n \leq n_\alpha} \|T^n\|\alpha^{-n_\alpha}$, which clearly implies $\|T^n\| \to 0$. $\qquad\square$

An operator $T \in \mathcal{B}[\mathcal{H}]$ on a complex Hilbert space $\mathcal{H}$ is *strongly stable* or *weakly stable* if the power sequence $\{T^n\}$ converges strongly or weakly to the null operator (i.e., if $\|T^n x\| \to 0$ for every x in $\mathcal{H}$, or $\langle T^n x\,;y\rangle \to 0$ for every x and y in $\mathcal{H}$ — equivalently, $\langle T^n x\,;x\rangle \to 0$ for every x in $\mathcal{H}$ — cf. Section 1.1). These are denoted by $T^n \xrightarrow{s} O$ and $T^n \xrightarrow{w} O$, respectively. Therefore, from what we have considered so far,

$$r(T) < 1 \quad\Longleftrightarrow\quad T^n \xrightarrow{u} O \quad\Longrightarrow$$

$$T^n \xrightarrow{s} O \quad\Longrightarrow\quad T^n \xrightarrow{w} O \quad\Longrightarrow\quad \sup_{n} \|T^n\| < \infty \quad\Longrightarrow\quad r(T) \leq 1.$$

The converses to the above one-way implications fail in general. The next result applies the preceding characterization of uniform stability to extend the Neumann expansion of Theorem 1.3.

Corollary 2.12. *Let $T \in \mathcal{B}[\mathcal{X}]$ be an operator on a complex Banach space, and let $\lambda \in \mathbb{C}$ be any nonzero complex number.*

(a) *$r(T) < |\lambda|$ if and only if $\left\{\sum_{k=0}^{n} \left(\frac{T}{\lambda}\right)^k\right\}$ converges uniformly. In this case we get $\lambda \in \rho(T)$ and $R_T(\lambda) = (\lambda I - T)^{-1} = \frac{1}{\lambda} \sum_{k=0}^{\infty} \left(\frac{T}{\lambda}\right)^k$ where $\sum_{k=0}^{\infty} \left(\frac{T}{\lambda}\right)^k$ denotes the uniform limit of $\left\{\sum_{k=0}^{n} \left(\frac{T}{\lambda}\right)^k\right\}$.*

(b) *If $r(T) = |\lambda|$ and $\left\{\sum_{k=0}^{n} \left(\frac{T}{\lambda}\right)^k\right\}$ converges strongly, then $\lambda \in \rho(T)$ and $R_T(\lambda) = (\lambda I - T)^{-1} = \frac{1}{\lambda} \sum_{k=0}^{\infty} \left(\frac{T}{\lambda}\right)^k$ where $\sum_{k=0}^{\infty} \left(\frac{T}{\lambda}\right)^k$ denotes the strong limit of $\left\{\sum_{k=0}^{n} \left(\frac{T}{\lambda}\right)^k\right\}$.*

(c) *If $|\lambda| < r(T)$, then $\left\{\sum_{k=0}^{n} \left(\frac{T}{\lambda}\right)^k\right\}$ does not converge strongly.*

Proof. If $\left\{\sum_{k=0}^{n} \left(\frac{T}{\lambda}\right)^k\right\}$ converges uniformly, then $\left(\frac{T}{\lambda}\right)^n \overset{u}{\longrightarrow} O$, and therefore $|\lambda|^{-1} r(T) = r\left(\frac{T}{\lambda}\right) < 1$ by Corollary 2.11. On the other hand, if $r(T) < |\lambda|$, then $\lambda \in \rho(T)$ so that $\lambda I - T \in \mathcal{G}[\mathcal{X}]$, and also $r\left(\frac{T}{\lambda}\right) = |\lambda|^{-1} r(T) < 1$. Hence, $\left\|\left(\frac{T}{\lambda}\right)^n\right\| \leq \beta \alpha^n$ for every $n \geq 0$, for some $\beta \geq 1$ and $\alpha \in (0,1)$, according to Corollary 2.11, and so $\sum_{k=0}^{\infty} \left\|\left(\frac{T}{\lambda}\right)^k\right\| < \infty$, which means that $\left\{\left(\frac{T}{\lambda}\right)^n\right\}$ is an absolutely summable sequence in $\mathcal{B}[\mathcal{X}]$. Now follow the steps in the proof of Theorem 1.3 to conclude the results in (a). If $\left\{\sum_{k=0}^{n} \left(\frac{T}{\lambda}\right)^k\right\}$ converges strongly, then $\left(\frac{T}{\lambda}\right)^n x \to 0$ in $\mathcal{X}$ for every $x \in \mathcal{X}$ so that $\sup_n \left\|\left(\frac{T}{\lambda}\right)^n x\right\| < \infty$ for every $x \in \mathcal{X}$, and hence $\sup_n \left\|\left(\frac{T}{\lambda}\right)^n\right\| < \infty$ (by the Banach–Steinhaus Theorem). Thus $|\lambda|^{-1} r(T) = r(\frac{T}{\lambda}) \leq 1$, which proves (c). Moreover,

$$(\lambda I - T)\frac{1}{\lambda}\sum_{k=0}^{n} \left(\tfrac{T}{\lambda}\right)^k = \frac{1}{\lambda}\sum_{k=0}^{n} \left(\tfrac{T}{\lambda}\right)^k (\lambda I - T) = I - \left(\tfrac{T}{\lambda}\right)^{n+1} \overset{s}{\longrightarrow} I.$$

Therefore, $(\lambda I - T)^{-1} = \frac{1}{\lambda} \sum_{k=0}^{\infty} \left(\frac{T}{\lambda}\right)^k$, where $\sum_{k=0}^{\infty} \left(\frac{T}{\lambda}\right)^k \in \mathcal{B}[\mathcal{X}]$ is the strong limit of $\left\{\sum_{k=0}^{n} \left(\frac{T}{\lambda}\right)^k\right\}$, which concludes the proof of (b). $\square$

2.5 Numerical Radius

The *numerical range* $W(T)$ of an operator $T \in \mathcal{B}[\mathcal{H}]$ acting on a nonzero complex Hilbert space $\mathcal{H}$ is the (nonempty) set consisting of the inner products $\langle Tx \, ; x \rangle$ for unit vectors $x \in \mathcal{H}$; that is,

$$W(T) = \big\{\lambda \in \mathbb{C}: \ \lambda = \langle Tx \, ; x \rangle \ \text{for some} \ \|x\| = 1\big\}.$$

It can be shown that $W(T)$ is a convex set in $\mathbb{C}$ (see, e.g., [50, Problem 210]), and it is clear that

$$W(T^*) = W(T)^*.$$

Theorem 2.13. $\sigma_P(T) \cup \sigma_R(T) \subseteq W(T)$ *and* $\sigma(T) \subseteq W(T)^-$.

Proof. Take any operator $T \in \mathcal{B}[\mathcal{H}]$ on a nonzero complex Hilbert space $\mathcal{H}$. If $\lambda \in \sigma_P(T)$, then there is a unit vector $x \in \mathcal{H}$ such that $Tx = \lambda x$. Hence $\langle Tx \,;\, x \rangle = \lambda \|x\|^2 = \lambda$ so that $\lambda \in W(T)$. If $\lambda \in \sigma_R(T)$, then $\overline{\lambda} \in \sigma_P(T^*)$ by Theorem 2.6, and so $\overline{\lambda} \in W(T^*)$. Thus $\lambda \in W(T)$. Therefore,

$$\sigma_P(T) \cup \sigma_R(T) \subseteq W(T).$$

If $\lambda \in \sigma_{AP}(T)$, then there is a sequence $\{x_n\}$ of unit vectors in $\mathcal{H}$ such that $\|(\lambda I - T)x_n\| \to 0$ by Theorem 2.4. Hence

$$0 \leq |\lambda - \langle Tx_n \,;\, x_n \rangle| = |\langle (\lambda I - T)x_n \,;\, x_n \rangle| \leq \|(\lambda I - T)x_n\| \to 0$$

so that $\langle Tx_n\, x_n \rangle \to \lambda$. Since each $\langle Tx_n \,;\, x_n \rangle$ lies in $W(T)$, the classical Closed Set Theorem says that $\lambda \in W(T)^-$. Thus $\sigma_{AP}(T) \subseteq W(T)^-$, and so

$$\sigma(T) = \sigma_R(T) \cup \sigma_{AP}(T) \subseteq W(T)^-. \qquad \square$$

The *numerical radius* of an operator $T \in \mathcal{B}[\mathcal{H}]$ on a nonzero complex Hilbert space $\mathcal{H}$ is the nonnegative number

$$w(T) = \sup_{\lambda \in W(T)} |\lambda| = \sup_{\|x\|=1} |\langle Tx \,;\, x \rangle|.$$

It is readily verified that

$$w(T^*) = w(T) \quad \text{and} \quad w(T^*T) = \|T\|^2.$$

Unlike the spectral radius, the numerical radius is a norm on $\mathcal{B}[\mathcal{H}]$. That is, $0 \leq w(T)$ for every $T \in \mathcal{B}[\mathcal{H}]$ and $0 < w(T)$ if $T \neq O$, $w(\alpha T) = |\alpha| w(T)$, and $w(T + S) \leq w(T) + w(S)$ for every $\alpha \in \mathbb{C}$ and every $S, T \in \mathcal{B}[\mathcal{H}]$. However, the numerical radius does not have the operator norm property in the sense that the inequality $w(ST) \leq w(S)w(T)$ is *not* true for all operators $S, T \in \mathcal{B}[\mathcal{H}]$. Nevertheless, the *power inequality* holds: $w(T^n) \leq w(T)^n$ for all $T \in \mathcal{B}[\mathcal{H}]$ and every positive integer n (see, e.g., [50, p. 118 and Problem 221]). Moreover, the numerical radius is a norm equivalent to the (induced uniform) operator norm of $\mathcal{B}[\mathcal{H}]$ and dominates the spectral radius, as in the next theorem.

Theorem 2.14. $0 \leq r(T) \leq w(T) \leq \|T\| \leq 2w(T)$.

Proof. Take any $T \in \mathcal{B}[\mathcal{H}]$. Since $\sigma(T) \subseteq W(T)^-$, we get $r(T) \leq w(T)$. Moreover, $w(T) = \sup_{\|x\|=1} |\langle Tx \,;\, x \rangle| \leq \sup_{\|x\|=1} \|Tx\| = \|T\|$. Now recall that, by the polarization identity (cf. Proposition 1.A),

$$\langle Tx \,;\, y \rangle = \tfrac{1}{4}\big(\langle T(x + y) \,;\, (x + y) \rangle - \langle T(x - y) \,;\, (x - y) \rangle$$
$$+ \, i\langle T(x + iy) \,;\, (x + iy) \rangle - i\langle T(x - iy) \,;\, (x - iy) \rangle\big)$$

for every x, y in $\mathcal{H}$. Therefore, since $|\langle Tz\,;z\rangle| \leq \sup_{\|u\|=1} |\langle Tu\,;u\rangle| \|z\|^2 = w(T)\|z\|^2$ for every $z \in \mathcal{H}$, it follows that, for every $x, y \in \mathcal{H}$,

$$\begin{aligned}
|\langle Tx\,;y\rangle| &\leq \tfrac{1}{4}\big(|\langle T(x+y)\,;(x+y)\rangle| + |\langle T(x-y)\,;(x-y)\rangle| \\
&\quad + |\langle T(x+iy)\,;(x+iy)\rangle| + |\langle T(x-iy)\,;(x-iy)\rangle|\big) \\
&\leq \tfrac{1}{4} w(T)\big(\|x+y\|^2 + \|x-y\|^2 + \|x+iy\|^2 + \|x-iy\|^2\big).
\end{aligned}$$

So it follows by the parallelogram law (cf. Proposition 1.A) that

$$|\langle Tx\,;y\rangle| \leq w(T)\big(\|x\|^2 + \|y\|^2\big) \leq 2w(T)$$

whenever $\|x\| = \|y\| = 1$. Thus, since $\|T\| = \sup_{\|x\|=\|y\|=1} |\langle Tx\,;y\rangle|$ (see, e.g., [66, Corollary 5.71]), it follows that $\|T\| \leq 2w(T)$. $\qquad\square$

An operator $T \in \mathcal{B}[\mathcal{H}]$ is *spectraloid* if $r(T) = w(T)$. Recall that an operator is normaloid if $r(T) = \|T\|$ or, equivalently, if $\|T^n\| = \|T\|^n$ for every $n \geq 1$ (see Theorem 1.11). The next result is a straightforward application of the previous theorem.

Corollary 2.15. *Every normaloid operator is spectraloid.*

Indeed, $r(T) = \|T\|$ implies $r(T) = w(T)$, but $r(T) = \|T\|$ also implies $w(T) = \|T\|$ (according to Theorem 2.14). Thus $w(T) = \|T\|$ is a property of every normaloid operator on $\mathcal{H}$. Actually, this can be viewed as a third definition of a normaloid operator on a complex Hilbert space.

Theorem 2.16. $T \in \mathcal{B}[\mathcal{H}]$ *is normaloid if and only if* $w(T) = \|T\|$.

Proof. Half of the proof was presented above. It remains to prove that

$$w(T) = \|T\| \quad \text{implies} \quad r(T) = \|T\|.$$

Suppose $w(T) = \|T\|$ (and $T \neq O$ to avoid trivialities). Recall that $W(T)^-$ is compact in $\mathbb{C}$ (for $W(T)$ is clearly bounded). Thus $\max_{\lambda \in W(T)^-} |\lambda| = \sup_{\lambda \in W(T)^-} |\lambda| = \sup_{\lambda \in W(T)} |\lambda| = w(T) = \|T\|$, and therefore there exists $\lambda \in W(T)^-$ such that $\lambda = \|T\|$. Since $W(T)$ is nonempty, λ is a point of adherence of $W(T)$, and so there is a sequence $\{\lambda_n\}$ with each λ_n in $W(T)$ such that $\lambda_n \to \lambda$. This means that there is a sequence $\{x_n\}$ of unit vectors in $\mathcal{H}$ (i.e., $\|x_n\| = 1$) such that $\lambda_n = \langle Tx_n\,;x_n\rangle \to \lambda$, where $|\lambda| = \|T\| \neq 0$. Hence, if $S = \lambda^{-1}T \in \mathcal{B}[\mathcal{H}]$, then

$$\langle Sx_n\,;x_n\rangle \to 1.$$

Claim. $\|Sx_n\| \to 1$ and $\operatorname{Re}\langle Sx_n\,;x_n\rangle \to 1$.

Proof. $|\langle Sx_n\,;x_n\rangle| \leq \|Sx_n\| \leq \|S\| = 1$ for each n. But $\langle Sx_n\,;x_n\rangle \to 1$ implies $|\langle Sx_n\,;x_n\rangle| \to 1$ (and so $\|Sx_n\| \to 1$) and also $\operatorname{Re}\langle Sx_n\,;x_n\rangle \to 1$ (since $|\cdot|$ and $\operatorname{Re}(\cdot)$ are continuous functions), which concludes the proof.

Then $\|(I-S)x_n\|^2 = \|Sx_n - x_n\|^2 = \|Sx_n\|^2 - 2\mathrm{Re}\langle Sx_n\,;x_n\rangle + \|x_n\|^2 \to 0$ so that $1 \in \sigma_{AP}(S) \subseteq \sigma(S)$ (cf. Theorem 2.4). Hence $r(S) \geq 1$, and so $r(T) = r(\lambda S) = |\lambda|\,r(S) \geq |\lambda| = \|T\|$, which implies that $r(T) = \|T\|$ (because $r(T) \leq \|T\|$ for every operator T). $\qquad\square$

Remark. If $T \in \mathcal{B}[\mathcal{H}]$ is spectraloid and quasinilpotent, then $T = O$. In fact, if $w(T) = 0$, then $T = O$ (since the numerical radius is a norm — also see Theorem 2.14); in particular, if $w(T) = r(T) = 0$, then $T = O$. Therefore, *the unique normal (or hyponormal, or normaloid, or spectraloid) quasinilpotent operator is the null operator*.

Corollary 2.17. *If there exists $\lambda \in W(T)$ such that $|\lambda| = \|T\|$, then T is normaloid and $\lambda \in \sigma_P(T)$. In other words, if there exists a unit vector x such that $\|T\| = |\langle Tx\,;x\rangle|$, then $r(T) = w(T) = \|T\|$ and $\langle Tx\,;x\rangle \in \sigma_P(T)$.*

Proof. If $\lambda \in W(T)$ is such that $|\lambda| = \|T\|$, then $w(T) = \|T\|$ (Theorem 2.14) so that T is normaloid (Theorem 2.16). Moreover, since $\lambda = \langle Tx\,;x\rangle$ for some unit vector x, it follows that $\|T\| = |\lambda| = |\langle Tx\,;x\rangle| \leq \|Tx\|\|x\| \leq \|T\|$, and hence $|\langle Tx\,;x\rangle| = \|Tx\|\|x\|$. That is, the Schwarz inequality becomes an identity, which implies that $Tx = \alpha x$ for some $\alpha \in \mathbb{C}$ (see, e.g., [66, Problem 5.2]). Thus $\alpha \in \sigma_P(T)$. But $\alpha = \alpha\|x\|^2 = \langle \alpha x\,;x\rangle = \langle Tx\,;x\rangle = \lambda$. $\qquad\square$

2.6 Spectrum of Compact Operators

The spectral theory of compact operators plays a central role in the Spectral Theorem for compact normal operators of the next chapter. Normal operators were defined on Hilbert spaces; thus we keep on working with compact operators on Hilbert spaces, as we did in Section 1.8, although the spectral theory of compact operators can be equally developed on nonzero complex Banach spaces. So we assume that all operators in this section act on a nonzero complex Hilbert space $\mathcal{H}$. The main result for characterizing the spectrum of compact operators is the Fredholm Alternative of Corollary 1.20, which can be restated as follows.

Theorem 2.18. (FREDHOLM ALTERNATIVE). *Take $T \in \mathcal{B}_\infty[\mathcal{H}]$. If $\lambda \in \mathbb{C}\backslash\{0\}$, then $\lambda \in \rho(T) \cup \sigma_P(T)$. Equivalently,*

$$\sigma(T)\backslash\{0\} = \sigma_P(T)\backslash\{0\}.$$

Moreover, if $\lambda \in \mathbb{C}\backslash\{0\}$, then $\dim \mathcal{N}(\lambda I - T) = \dim \mathcal{N}(\overline{\lambda}I - T^*) < \infty$ so that $\lambda \in \rho(T) \cup \sigma_{P_4}(T)$. Equivalently,

$$\sigma(T)\backslash\{0\} = \sigma_{P_4}(T)\backslash\{0\}.$$

Proof. Take a compact operator T on a Hilbert space $\mathcal{H}$ and a nonzero scalar λ in $\mathbb{C}$. Corollary 1.20 and the diagram of Section 2.2 ensure that

$$\lambda \in \rho(T) \cup \sigma_{P_1}(T) \cup \sigma_{R_1}(T) \cup \sigma_{P_4}(T).$$

Also by Corollary 1.20, $\mathcal{N}(\lambda I - T) = \{0\}$ if and only if $\mathcal{N}(\overline{\lambda}I - T^*) = \{0\}$, so that $\lambda \in \sigma_P(T)$ if and only if $\overline{\lambda} \in \sigma_P(T^*)$. Thus, $\lambda \notin \sigma_{P_1}(T) \cup \sigma_{R_1}(T)$ by Theorem 2.6, so that $\lambda \in \rho(T) \cup \sigma_{P_4}(T)$ or, equivalently, $\lambda \in \rho(T) \cup \sigma_P(T)$ (since $\lambda \notin \sigma_{P_1}(T) \cup \sigma_{P_2}(T) \cup \sigma_{P_3}(T)$). Therefore,

$$\sigma(T)\backslash\{0\} = \sigma_P(T)\backslash\{0\} = \sigma_{P_4}(T)\backslash\{0\}. \qquad \square$$

The scalar 0 may be anywhere. That is, if $T \in \mathcal{B}_\infty[\mathcal{H}]$, then $\lambda = 0$ may lie in $\sigma_P(T)$, $\sigma_R(T)$, $\sigma_C(T)$, or $\rho(T)$. However, if T is a compact operator on a nonzero space $\mathcal{H}$ and $0 \in \rho(T)$, then $\mathcal{H}$ must be finite dimensional. Indeed, if $0 \in \rho(T)$, then $T^{-1} \in \mathcal{B}[\mathcal{H}]$ so that $I = T^{-1}T$ is compact (since $\mathcal{B}_\infty[\mathcal{H}]$ is an ideal of $\mathcal{B}[\mathcal{H}]$), which implies that $\mathcal{H}$ is finite dimensional (cf. Proposition 1.Y). The preceding theorem in fact is a rewriting of the Fredholm Alternative (and it is also referred to as the Fredholm Alternative). It will be applied often from now on. Here is a first application. Let $\mathcal{B}_0[\mathcal{H}]$ denote the class of all *finite-rank operators* on $\mathcal{H}$ (i.e., the class of all operators from $\mathcal{B}[\mathcal{H}]$ with a finite-dimensional range). Recall that $\mathcal{B}_0[\mathcal{H}] \subseteq \mathcal{B}_\infty[\mathcal{H}]$ (finite-rank operators are compact — cf. Proposition 1.X). Let $\#A$ denote the cardinality of a set A, so that $\#A < \infty$ means "A is a finite set".

Corollary 2.19. If $T \in \mathcal{B}_0[\mathcal{H}]$, then

$$\sigma(T) = \sigma_P(T) = \sigma_{P_4}(T) \quad \text{and} \quad \#\sigma(T) < \infty.$$

Proof. If $\dim \mathcal{H} < \infty$, then an injective operator is surjective, and linear manifolds are closed (see, e.g., [66, Problem 2.18 and Corollary 4.29]), and so the diagram of Section 2.2 says that $\sigma(T) = \sigma_P(T) = \sigma_{P_4}(T)$ (for $\sigma_{P_1}(T) = \sigma_{R_1}(T^*)^*$ according to Theorem 2.6). On the other hand, suppose $\dim \mathcal{H} = \infty$. Since $\mathcal{B}_0[\mathcal{H}] \subseteq \mathcal{B}_\infty[\mathcal{H}]$, Theorem 2.18 says that

$$\sigma(T)\backslash\{0\} = \sigma_P(T)\backslash\{0\} = \sigma_{P_4}(T)\backslash\{0\}.$$

Since $\dim \mathcal{R}(T) < \infty$ and $\dim \mathcal{H} = \infty$, it follows that $\mathcal{R}(T)^- = \mathcal{R}(T) \neq \mathcal{H}$ and $\mathcal{N}(T) \neq \{0\}$ (because $\dim \mathcal{N}(T) + \dim \mathcal{R}(T) = \dim \mathcal{H}$; see, e.g., [66, Problem 2.17]). Then $0 \in \sigma_{P_4}(T)$ (cf. diagram of Section 2.2), and therefore

$$\sigma(T) = \sigma_P(T) = \sigma_{P_4}(T).$$

If $\sigma_P(T)$ is infinite, then there exists an infinite set of linearly independent eigenvectors of T (Theorem 2.3). Since every eigenvector of T lies in $\mathcal{R}(T)$, this implies that $\dim \mathcal{R}(T) = \infty$ (because every linearly independent subset of a linear space is included in some Hamel bases — see, e.g., [66, Theorem 2.5]), which is a contradiction. Conclusion: $\sigma_P(T)$ must be finite. $\qquad \square$

In particular, the above result clearly holds if $\mathcal{H}$ is finite dimensional since, as we saw above, $\dim \mathcal{H} < \infty$ implies $\mathcal{B}[\mathcal{H}] = \mathcal{B}_0[\mathcal{H}]$.

Corollary 2.20. *Take an arbitrary compact operator $T \in \mathcal{B}_\infty[\mathcal{H}]$.*

(a) *0 is the only possible accumulation point of $\sigma(T)$.*

(b) *If $\lambda \in \sigma(T)\backslash\{0\}$, then λ is an isolated point of $\sigma(T)$.*

(c) *$\sigma(T)\backslash\{0\}$ is a discrete subset of $\mathbb{C}$.*

(d) *$\sigma(T)$ is countable.*

Proof. Let T be a compact operator on $\mathcal{H}$.

Claim. An infinite sequence of distinct points of $\sigma(T)$ converges to zero.

Proof. Let $\{\lambda_n\}_{n=1}^\infty$ be an infinite sequence of distinct points of $\sigma(T)$. Without loss of generality, suppose that every λ_n is nonzero. Since T is compact and $0 \neq \lambda_n \in \sigma(T)$, it follows by Theorem 2.18 that $\lambda_n \in \sigma_P(T)$. Let $\{x_n\}_{n=1}^\infty$ be a sequence of eigenvectors associated with $\{\lambda_n\}_{n=1}^\infty$ (i.e., $Tx_n = \lambda_n x_n$ with each $x_n \neq 0$), which is a sequence of linearly independent vectors by Theorem 2.3. For each $n \geq 1$, set

$$\mathcal{M}_n = \mathrm{span}\{x_i\}_{i=1}^n,$$

which is a subspace of $\mathcal{H}$ with $\dim \mathcal{M}_n = n$, and

$$\mathcal{M}_n \subset \mathcal{M}_{n+1}$$

for every $n \geq 1$ (because $\{x_i\}_{i=1}^{n+1}$ is linearly independent and so x_{n+1} lies in $\mathcal{M}_{n+1}\backslash\mathcal{M}_n$). From now on the argument is similar to that in the proof of Theorem 1.18. Since each $\mathcal{M}_n$ is a proper subspace of the Hilbert space $\mathcal{M}_{n+1}$, it follows that there exists y_{n+1} in $\mathcal{M}_{n+1}$ with $\|y_{n+1}\| = 1$ for which $\frac{1}{2} < \inf_{u \in \mathcal{M}_n} \|y_{n+1} - u\|$. Write $y_{n+1} = \sum_{i=1}^{n+1} \alpha_i x_i$ in $\mathcal{M}_{n+1}$ so that

$$(\lambda_{n+1}I - T)y_{n+1} = \sum_{i=1}^{n+1} \alpha_i(\lambda_{n+1} - \lambda_i)x_i = \sum_{i=1}^n \alpha_i(\lambda_{n+1} - \lambda_i)x_i \in \mathcal{M}_n.$$

Recall that $\lambda_n \neq 0$ for all n, take any pair of integers $1 \leq m < n$, and set

$$y = y_m - \lambda_m^{-1}(\lambda_m I - T)y_m + \lambda_n^{-1}(\lambda_n I - T)y_n$$

so that $T(\lambda_m^{-1}y_m) - T(\lambda_n^{-1}y_n) = y - y_n$. Since y lies in $\mathcal{M}_{n-1}$,

$$\tfrac{1}{2} < \|y - y_n\| = \|T(\lambda_m^{-1}y_m) - T(\lambda_n^{-1}y_n)\|,$$

which implies that the sequence $\{T(\lambda_n^{-1}y_n)\}$ has no convergent subsequence. Thus, since T is compact, Proposition 1.S ensures that $\{\lambda_n^{-1}y_n\}$ has no bounded subsequence. That is, $\sup_k |\lambda_k|^{-1} = \sup_k \|\lambda_k^{-1}y_k\| = \infty$, and so $\inf_k |\lambda_k| = 0$ for every subsequence $\{\lambda_k\}_{k=1}^\infty$ of $\{\lambda_n\}_{n=1}^\infty$. Thus $\lambda_n \to 0$, which concludes the proof of the claimed result.

(a) Thus, if $\lambda \neq 0$, then there is no sequence of distinct points in $\sigma(T)$ that converges to λ; that is, $\lambda \neq 0$ is not an accumulation point of $\sigma(T)$.

(b) Therefore, every λ in $\sigma(T)\backslash\{0\}$ is not an accumulation point of $\sigma(T)$; equivalently, every λ in $\sigma(T)\backslash\{0\}$ is an isolated point of $\sigma(T)$.

(c) Hence $\sigma(T)\backslash\{0\}$ consists entirely of isolated points, which means that $\sigma(T)\backslash\{0\}$ is a discrete subset of $\mathbb{C}$.

(d) Since *a discrete subset of a separable metric space is countable* (see, e.g., [66, Example 3.Q]), and since $\mathbb{C}$ is separable, $\sigma(T)\backslash\{0\}$ is countable. $\square$

Corollary 2.21. *If an operator $T \in \mathcal{B}[\mathcal{H}]$ is compact and normaloid, then $\sigma_P(T) \neq \varnothing$ and there exists $\lambda \in \sigma_P(T)$ such that $|\lambda| = \|T\|$.*

Proof. Suppose T is normaloid (i.e., $r(T) = \|T\|$). Thus $\sigma(T) = \{0\}$ only if $T = O$. If $T = O$ and $\mathcal{H} \neq \{0\}$, then $0 \in \sigma_P(T)$ and $\|T\| = 0$. If $T \neq O$, then $\sigma(T) \neq \{0\}$ and $\|T\| = r(T) = \max_{\lambda \in \sigma(T)} |\lambda|$, so that there exists $\lambda \neq 0$ in $\sigma(T)$ such that $|\lambda| = \|T\|$. Moreover, if T is compact and $\sigma(T) \neq \{0\}$, then $\varnothing \neq \sigma(T)\backslash\{0\} \subseteq \sigma_P(T)$ by Theorem 2.18. Hence $r(T) = \max_{\lambda \in \sigma(T)} |\lambda| = \max_{\lambda \in \sigma_P(T)} |\lambda| = \|T\|$. Thus there exists $\lambda \in \sigma_P(T)$ such that $|\lambda| = \|T\|$. $\square$

Corollary 2.22. *Every compact hyponormal operator is normal.*

Proof. Suppose $T \in \mathcal{B}[\mathcal{H}]$ is a compact hyponormal operator on a nonzero complex Hilbert space $\mathcal{H}$. Corollary 2.21 says that $\sigma_P(T) \neq \varnothing$. Consider the subspace $\mathcal{M} = \left(\sum_{\lambda \in \sigma_P(T)} \mathcal{N}(\lambda I - T)\right)^-$ of Theorem 1.16 with $\{\lambda_\gamma\}_{\gamma \in \Gamma} = \sigma_P(T)$. Observe that $\sigma_P(T|_{\mathcal{M}^\perp}) = \varnothing$. Indeed, if there is a $\lambda \in \sigma_P(T|_{\mathcal{M}^\perp})$, then there exists $0 \neq x \in \mathcal{M}^\perp$ such that $\lambda x = T|_{\mathcal{M}^\perp} x = Tx$, and so $x \in \mathcal{N}(\lambda I - T) \subseteq \mathcal{M}$, which is a contradiction. Moreover, recall that $T|_{\mathcal{M}^\perp}$ is compact and hyponormal (Propositions 1.O and 1.U). Thus, if $\mathcal{M}^\perp \neq \{0\}$, then Corollary 2.21 says that $\sigma_P(T|_{\mathcal{M}^\perp}) \neq \varnothing$, which is another contradiction. Therefore, $\mathcal{M}^\perp = \{0\}$ so that $\mathcal{M} = \mathcal{H}$ (see Section 1.3), and hence $T = T|_{\mathcal{H}} = T|_{\mathcal{M}}$ is normal according to Theorem 1.16. $\square$

2.7 Additional Propositions

Proposition 2.A. *Let $\mathcal{H} \neq \{0\}$ be a complex Hilbert space and let $\mathbb{T}$ denote the unit circle about the origin of the complex plane.*

(a) *If $H \in \mathcal{B}[\mathcal{H}]$ is hyponormal, then $\sigma_P(H)^* \subseteq \sigma_P(H^*)$ and $\sigma_R(H^*) = \varnothing$.*

(b) *If $N \in \mathcal{B}[\mathcal{H}]$ is normal, then $\sigma_P(N^*) = \sigma_P(N)^*$ and $\sigma_R(N) = \varnothing$.*

(c) *If $U \in \mathcal{B}[\mathcal{H}]$ is unitary, then $\sigma(U) \subseteq \mathbb{T}$.*

(d) *If $A \in \mathcal{B}[\mathcal{H}]$ is self-adjoint, then $\sigma(A) \subset \mathbb{R}$.*

(e) *If $Q \in \mathcal{B}[\mathcal{H}]$ is nonnegative, then $\sigma(Q) \subset [0, \infty)$.*

(f) *If $R \in \mathcal{B}[\mathcal{H}]$ is strictly positive, then $\sigma(R) \subset [\alpha, \infty)$ for some $\alpha > 0$.*

(g) *If $E \in \mathcal{B}[\mathcal{H}]$ is a nontrivial projection, then $\sigma(E) = \sigma_P(E) = \{0, 1\}$.*

Proposition 2.B. *Similarity preserves the spectrum and its parts, and so it preserves the spectral radius. That is, let $\mathcal{H}$ and $\mathcal{K}$ be nonzero complex Hilbert spaces. For every $T \in \mathcal{B}[\mathcal{H}]$ and $W \in \mathcal{G}[\mathcal{H}, \mathcal{K}]$,*

(a) $\sigma_P(T) = \sigma_P(WTW^{-1})$,

(b) $\sigma_R(T) = \sigma_R(WTW^{-1})$,

(c) $\sigma_C(T) = \sigma_C(WTW^{-1})$.

Hence $\sigma(T) = \sigma(WTW^{-1})$, $\rho(T) = \rho(WTW^{-1})$, and $r(T) = r(WTW^{-1})$. Unitary equivalence also preserves the norm: if W is a unitary transformation, then, in addition, $\|T\| = \|WTW^{-1}\|$.

Proposition 2.C. $\sigma(ST)\backslash\{0\} = \sigma(TS)\backslash\{0\}$ *for every $S, T \in \mathcal{B}[\mathcal{H}]$.*

Proposition 2.D. *If $Q \in \mathcal{B}[\mathcal{H}]$ is nonnegative, then $\sigma(Q^{\frac{1}{2}}) = \sigma(Q)^{\frac{1}{2}}$.*

Proposition 2.E. *Take any operator $T \in \mathcal{B}[\mathcal{H}]$. Let d denote the usual distance in $\mathbb{C}$. If $\lambda \in \rho(T)$, then*

$$r((\lambda I - T)^{-1}) = [d(\lambda, \sigma(T))]^{-1}.$$

If T is hyponormal and $\lambda \in \rho(T)$, then

$$\|(\lambda I - T)^{-1}\| = [d(\lambda, \sigma(T))]^{-1}.$$

Proposition 2.F. *Let $\{\mathcal{H}_k\}$ be a collection of Hilbert spaces, let $\{T_k\}$ be a (similarly indexed) collection of operators with each T_k in $\mathcal{B}[\mathcal{H}_k]$, and consider the (orthogonal) direct sum $\bigoplus_k T_k$ in $\mathcal{B}[\bigoplus_k \mathcal{H}_k]$. Then*

(a) $\sigma_P(\bigoplus_k T_k) = \bigcup_k \sigma_P(T_k)$,

(b) $\sigma(\bigoplus_k T_k) = \bigcup_k \sigma(T_k)$ *if the collection $\{T_k\}$ is finite.*

In general (if the collection $\{T_k\}$ is not finite), then

(c) $(\bigcup_k \sigma(T_k))^- \subseteq \sigma(\bigoplus_k T_k)$ *and the inclusion may be proper.*

However, if $\|(\lambda I - T_k)^{-1}\| = [d(\lambda, \sigma(T_k))]^{-1}$ for each k and every $\lambda \in \rho(T_k)$,

(d) $(\bigcup_k \sigma(T_k))^- = \sigma(\bigoplus_k T_k)$, *which happens whenever each T_k is hyponormal.*

Proposition 2.G. *An operator $T \in \mathcal{B}[\mathcal{X}]$ on a complex Banach space is normaloid if and only if there is a $\lambda \in \sigma(T)$ such that $|\lambda| = \|T\|$.*

Proposition 2.H. *For every operator $T \in \mathcal{B}[\mathcal{H}]$ on a complex Hilbert space,*

$$\sigma_R(T) \subseteq \big\{\lambda \in \mathbb{C}: |\lambda| < \|T\|\big\}.$$

Proposition 2.I. *If $\mathcal{H}$ and $\mathcal{K}$ are complex Hilbert spaces and $T \in \mathcal{B}[\mathcal{H}]$, then*

$$r(T) = \inf_{W \in \mathcal{G}[\mathcal{H},\mathcal{K}]} \|WTW^{-1}\|.$$

The spectral radius expression in the preceding proposition ensures that *an operator is uniformly stable if and only if it is similar to a strict contraction.*

Proposition 2.J. *If $T \in \mathcal{B}[\mathcal{X}]$ is a nilpotent operator on a complex Banach space, then $\sigma(T) = \sigma_P(T) = \{0\}$.*

Proposition 2.K. *An operator $T \in \mathcal{B}[\mathcal{H}]$ on a complex Hilbert space is spectraloid if and only if $w(T^n) = w(T)^n$ for every $n \geq 0$.*

An operator $T \in \mathcal{B}[\mathcal{H}]$ on a separable infinite-dimensional Hilbert space $\mathcal{H}$ is *diagonalizable* if $Tx = \sum_{k=0}^{\infty} \alpha_k \langle x\,;e_k\rangle e_k$ for every $x \in \mathcal{H}$, for some orthonormal basis $\{e_k\}_{k=0}^{\infty}$ for $\mathcal{H}$ and some bounded sequence $\{\alpha_k\}_{k=0}^{\infty}$ of scalars.

Proposition 2.L. *If $T \in \mathcal{B}[\mathcal{H}]$ is diagonalizable and $\mathcal{H}$ is complex, then*

$$\sigma_P(T) = \big\{\lambda \in \mathbb{C}\colon \lambda = \alpha_k \text{ for some } k \geq 1\big\}, \qquad \sigma_R(T) = \varnothing, \qquad \text{and}$$

$$\sigma_C(T) = \big\{\lambda \in \mathbb{C}\colon \inf_k |\lambda - \alpha_k| = 0 \text{ and } \lambda \neq \alpha_k \text{ for every } k \geq 1\big\}.$$

An operator $S_+ \in \mathcal{B}[\mathcal{K}_+]$ on a Hilbert space $\mathcal{K}_+$ is a *unilateral shift*, and an operator $S \in \mathcal{B}[\mathcal{K}]$ on a Hilbert space $\mathcal{K}$ is a *bilateral shift*, if there exists an infinite sequence $\{\mathcal{H}_k\}_{k=0}^{\infty}$ and an infinite family $\{\mathcal{H}_k\}_{k=-\infty}^{\infty}$ of nonzero pairwise orthogonal subspaces of $\mathcal{K}_+$ and $\mathcal{K}$ such that $\mathcal{K}_+ = \bigoplus_{k=0}^{\infty} \mathcal{H}_k$ and $\mathcal{K} = \bigoplus_{k=-\infty}^{\infty} \mathcal{H}_k$ (cf. Section 1.3), and both S_+ and S map each $\mathcal{H}_k$ isometrically onto $\mathcal{H}_{k+1}$, so that each transformation $U_{+(k+1)} = S_+|_{\mathcal{H}_k}\colon \mathcal{H}_k \to \mathcal{H}_{k+1}$, and each transformation $U_{k+1} = S|_{\mathcal{H}_k}\colon \mathcal{H}_k \to \mathcal{H}_{k+1}$, is unitary, and therefore $\dim \mathcal{H}_{k+1} = \dim \mathcal{H}_k$. Such a common dimension is the *multiplicity* of S_+ and S. If $\mathcal{H}_k = \mathcal{H}$ for all k, then $\mathcal{K}_+ = \ell_+^2(\mathcal{H}) = \bigoplus_{k=0}^{\infty} \mathcal{H}$ and $\mathcal{K} = \ell^2(\mathcal{H}) = \bigoplus_{k=-\infty}^{\infty} \mathcal{H}$ are the direct orthogonal sums of countably infinite copies of a single nonzero Hilbert space $\mathcal{H}$, indexed either by the nonnegative integers or by all integers, which are precisely the Hilbert spaces consisting of all square-summable $\mathcal{H}$-valued sequences $\{x_k\}_{k=0}^{\infty}$ and of all square-summable $\mathcal{H}$-valued families $\{x_k\}_{k=-\infty}^{\infty}$. In this case (if $\mathcal{H}_k = \mathcal{H}$ for all k), $U_{+(k+1)} = S_+|_{\mathcal{H}} = U_+$ and $U_{k+1} = S|_{\mathcal{H}} = U$ for all k, where U_+ and U are any unitary operators on $\mathcal{H}$. In particular, if $U_+ = U = I$, the identity on $\mathcal{H}$, then S_+ and S are referred to as the *canonical* unilateral and bilateral shifts on $\ell_+^2(\mathcal{H})$ and on $\ell^2(\mathcal{H})$. The adjoint $S_+^* \in \mathcal{B}[\mathcal{K}_+]$ of $S_+ \in \mathcal{B}[\mathcal{K}_+]$ and the adjoint of $S^* \in \mathcal{B}[\mathcal{K}]$ of $S \in \mathcal{B}[\mathcal{K}]$ are referred to as a *backward unilateral shift* and as a *backward bilateral shift*. Writing $\bigoplus_{k=0}^{\infty} x_k$ for $\{x_k\}_{k=0}^{\infty}$ in $\bigoplus_{k=0}^{\infty} \mathcal{H}_k$, and $\bigoplus_{k=-\infty}^{\infty} x_k$ for $\{x_k\}_{k=-\infty}^{\infty}$ in $\bigoplus_{k=0}^{\infty} \mathcal{H}_k$, it follows that $S_+\colon \mathcal{K}_+ \to \mathcal{K}_+$ and $S_+^*\colon \mathcal{K}_+ \to \mathcal{K}_+$, and $S\colon \mathcal{K} \to \mathcal{K}$ and $S^*\colon \mathcal{K} \to \mathcal{K}$, are given by the formulas

$$S_+x = 0 \oplus \bigoplus_{k=1}^{\infty} U_{+(k)}\, x_{k-1} \quad \text{and} \quad S_+^*x = \bigoplus_{k=0}^{\infty} U_{+(k+1)}^* x_{k+1}$$

for all $x = \bigoplus_{k=0}^{\infty} x_k$ in $\mathcal{K}_+ = \bigoplus_{k=0}^{\infty} \mathcal{H}_k$, with 0 being the origin of $\mathcal{H}_0$, where $U_{+(k+1)}$ is any unitary transformation of $\mathcal{H}_k$ onto $\mathcal{H}_{k+1}$ for each $k \geq 0$, and

$$Sx = \bigoplus_{k=-\infty}^{\infty} U_k\, x_{k-1} \quad \text{and} \quad S^*x = \bigoplus_{k=\infty}^{\infty} U_{k+1}^*\, x_{k+1}$$

for all $x = \bigoplus_{k=-\infty}^{\infty} x_k$ in $\mathcal{K} = \bigoplus_{k=-\infty}^{\infty} \mathcal{H}_k$, where, for each integer k, U_{k+1} is any unitary transformation of $\mathcal{H}_k$ onto $\mathcal{H}_{k+1}$. The spectrum of a bilateral shift is simpler than that of a unilateral shift, for bilateral shifts are unitary operators (i.e., besides being isometries as unilateral shifts are, bilateral shifts are normal too). For a full treatment on shifts on Hilbert spaces see [49].

Proposition 2.M. *Let $\mathbb{D}$ and $\mathbb{T} = \partial\mathbb{D}$ denote the open unit disk and the unit circle about the origin of the complex plane, respectively. If $S_+ \in \mathcal{B}[\mathcal{K}_+]$ is a unilateral shift and $S \in \mathcal{B}[\mathcal{K}]$ is a bilateral shift on complex spaces, then*

(a) $\sigma_P(S_+) = \sigma_R(S_+^*) = \varnothing, \quad \sigma_R(S_+) = \sigma_P(S_+^*) = \mathbb{D}, \quad \sigma_C(S_+) = \sigma_C(S_+^*) = \mathbb{T}.$

(b) $\sigma(S) = \sigma(S^*) = \sigma_C(S^*) = \sigma_C(S) = \mathbb{T}.$

A *unilateral weighted shift* $T_+ = S_+ D_+$ in $\mathcal{B}[\ell_+^2(\mathcal{H})]$ is the product of a canonical unilateral shift S_+ and a diagonal operator $D_+ = \bigoplus_{k=0}^{\infty} \alpha_k I$, both in $\mathcal{B}[\ell_+^2(\mathcal{H})]$, where $\{\alpha_k\}_{k=0}^{\infty}$ is a bounded sequence of scalars. A *bilateral weighted shift* $T = SD$ in $\mathcal{B}[\ell^2(\mathcal{H})]$ is the product of a canonical bilateral shift S and a diagonal operator $D = \bigoplus_{k=-\infty}^{\infty} \alpha_k I$, both in $\mathcal{B}[\ell^2(\mathcal{H})]$, where $\{\alpha_k\}_{k=-\infty}^{\infty}$ is a bounded family of scalars.

Proposition 2.N. *Let $T_+ \in \mathcal{B}[\ell_+^2(\mathcal{H})]$ be a unilateral weighted shift, and let $T \in \mathcal{B}[\ell^2(\mathcal{H})]$ be a bilateral weighted shift, where $\mathcal{H}$ is complex.*

(a) *If $\alpha_k \to 0$ as $|k| \to \infty$, then T_+ and T are compact and quasinilpotent.*

If, in addition, $\alpha_k \neq 0$ for all k, then

(b) $\sigma(T_+) = \sigma_R(T_+) = \sigma_{R_2}(T_+) = \{0\} \quad \text{and} \quad \sigma(T_+^*) = \sigma_P(T_+^*) = \sigma_{P_2}(T_+^*) = \{0\},$

(c) $\sigma(T) = \sigma_C(T) = \sigma_C(T^*) = \sigma(T^*) = \{0\}.$

Proposition 2.O. *Let $T \in \mathcal{B}[\mathcal{H}]$ be an operator on a complex Hilbert space and let $\mathbb{D}$ be the open unit disk about the origin of the complex plane.*

(a) *If $T^n \xrightarrow{w} O$, then $\sigma_P(T) \subseteq \mathbb{D}$.*

(b) *If T is compact and $\sigma_P(T) \subseteq \mathbb{D}$, then $T^n \xrightarrow{u} O$.*

(c) *The concepts of weak, strong, and uniform stabilities coincide for a compact operator on a complex Hilbert space.*

Proposition 2.P. *Take an operator $T \in \mathcal{B}[\mathcal{H}]$ on a complex Hilbert space and let $\mathbb{T} = \partial\mathbb{D}$ be the unit circle about the origin of the complex plane.*

(a) $T^n \xrightarrow{u} O$ *if and only if* $T^n \xrightarrow{w} O$ *and* $\sigma_C(T) \cap \mathbb{T} = \varnothing$.

(b) *If the continuous spectrum does not intersect the unit circle, then the concepts of weak, strong, and uniform stabilities coincide.*

The concepts of resolvent set $\rho(T)$ and spectrum $\sigma(T)$ of an operator T in the unital complex Banach algebra $\mathcal{B}[\mathcal{X}]$ as in Section 2.1, namely, $\rho(T) = \{\lambda \in \mathbb{C}\colon \lambda I - T$ has an inverse in $\mathcal{B}[\mathcal{X}]\}$ and $\sigma(T) = \mathbb{C}\backslash\rho(T)$, of course, hold in any unital complex Banach algebra $\mathcal{A}$, and so does the concept of spectral radius $r(T) = \sup_{\lambda \in \sigma(T)} |\lambda|$, where the Gelfand–Beurling formula of Theorem 2.10, namely, $r(T) = \lim_n \|T^n\|^{\frac{1}{n}}$, holds in any unital complex Banach algebra (whose proof is essentially the same as the proof of Theorem 2.10). Recall that a *component* of a set in a topological space is any maximal (in the inclusion ordering) connected subset of it. A *hole* of a compact set is any bounded component of its complement. Thus the holes of the spectrum $\sigma(T)$ are the bounded components of the resolvent set $\rho(T) = \mathbb{C}\backslash\sigma(T)$.

If $\mathcal{A}'$ is a closed unital subalgebra of a unital complex Banach algebra $\mathcal{A}$ (for instance, $\mathcal{A} = \mathcal{B}[\mathcal{X}]$ where $\mathcal{X}$ is a Banach space), and if $T \in \mathcal{A}'$, then let $\rho'(T)$ be the resolvent set of T with respect to $\mathcal{A}'$, let $\sigma'(T) = \mathbb{C}\backslash\rho'(T)$ be the spectrum of T with respect to $\mathcal{A}'$, and set $r'(T) = \sup_{\lambda \in \sigma'(T)} |\lambda|$. Recall that a *homomorphism* (or an *algebra homomorphism*) between two algebras is a linear transformation between them that also preserves product. Let $\mathcal{A}'$ be a maximal (in the inclusion ordering) commutative subalgebra of a unital complex Banach algebra $\mathcal{A}$ (i.e., a commutative subalgebra of $\mathcal{A}$ that is not included in any other commutative subalgebra of $\mathcal{A}$). Note that $\mathcal{A}'$ is trivially unital, and closed in $\mathcal{A}$ because the closure of a commutative subalgebra of a Banach algebra is again commutative since multiplication is continuous in $\mathcal{A}$. Consider the (unital complex commutative) Banach algebra $\mathbb{C}$ (of all complex numbers). Let $\widehat{\mathcal{A}'} = \{\Phi\colon \mathcal{A}' \to \mathbb{C}\colon \Phi$ is an homomorphism$\}$ stand for the collection of all algebra homomorphisms of $\mathcal{A}'$ onto $\mathbb{C}$.

Proposition 2.Q. *Let $\mathcal{A}$ be any unital complex Banach algebra (for instance, $\mathcal{A} = \mathcal{B}[\mathcal{X}]$). If $T \in \mathcal{A}'$, where $\mathcal{A}'$ is any closed unital subalgebra of $\mathcal{A}$, then*

(a) $\rho'(T) \subseteq \rho(T)$ *and* $r'(T) = r(T)$ *(invariance of the spectral radius). Hence* $\partial\sigma'(T) \subseteq \partial\sigma(T)$ *and* $\sigma(T) \subseteq \sigma'(T)$.

Thus $\sigma'(T)$ is obtained by adding to $\sigma(T)$ some holes of $\sigma(T)$.

(b) *If $\mathcal{A}'$ is a maximal commutative subalgebra of $\mathcal{A}$, then*

$$\sigma(T) = \big\{\Phi(T) \in \mathbb{C}\colon \Phi \in \widehat{\mathcal{A}'}\big\} \quad \text{for each} \quad T \in \mathcal{A}'.$$

Moreover, in this case,

$$\sigma(T) = \sigma'(T).$$

Proposition 2.R. *If $\mathcal{A}$ is a unital complex Banach algebra, and if S, T in $\mathcal{A}$ commute (i.e., if $S, T \in \mathcal{A}$ and $ST - TS$), then*

$$\sigma(S + T) \subseteq \sigma(S) + \sigma(T) \quad \text{and} \quad \sigma(ST) \subseteq \sigma(S) \cdot \sigma(T).$$

If $\mathcal{M}$ is an invariant subspace for T, then it may happen that $\sigma(T|_{\mathcal{M}}) \nsubseteq \sigma(T)$. Sample: every unilateral shift is the restriction of a bilateral shift to an invariant subspace (see, e.g., [62, Lemma 2.14]). However, if $\mathcal{M}$ reduces T, then $\sigma(T|_{\mathcal{M}}) \subseteq \sigma(T)$ by Proposition 2.F(b). The *full spectrum* of $T \in \mathcal{B}[\mathcal{H}]$ (notation: $\sigma(T)^{\#}$) is the union of $\sigma(T)$ and all bounded components of $\rho(T)$ (i.e., $\sigma(T)^{\#}$ is the union of $\sigma(T)$ and all holes of $\sigma(T)$).

Proposition 2.S. *If $\mathcal{M}$ is T-invariant, then $\sigma(T|_{\mathcal{M}}) \subseteq \sigma(T)^{\#}$.*

Proposition 2.T. *Let $T \in \mathcal{B}[\mathcal{H}]$ and $S \in \mathcal{B}[\mathcal{K}]$ be operators on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. If $\sigma(T) \cap \sigma(S) = \varnothing$, then for every bounded linear transformation $Y \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$ there exists a unique bounded linear transformation $X \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$ such that $XT - SX = Y$. In particular,*

$$\sigma(T) \cap \sigma(S) = \varnothing \ \ and \ \ XT = SX \quad \Longrightarrow \quad X = O.$$

This is the Rosenblum Corollary, which will be used to prove the Fuglede Theorems in Chapter 3.

Notes: Again, as in the previous chapter, these are basic results that will be needed throughout the text. We will not point out here the original sources but, instead, well-known secondary sources where the reader can find the proofs for those propositions, as well as some deeper discussions on them. Proposition 2.A holds independently of the forthcoming Spectral Theorem (Theorem 3.11) — see, e.g., [66, Corollary 6.18]. A partial converse, however, needs the Spectral Theorem (see Proposition 3.D in Chapter 3). Proposition 2.B is a standard result (see, e.g., [50, Problem 75] and [66, Problem 6.10]), as is Proposition 2.C (see, e.g., [50, Problem 76]). Proposition 2.D also dismisses the Spectral Theorem of the next chapter; it is obtained by the Square Root Theorem (Proposition 1.M) and by the Spectral Mapping Theorem (Theorem 2.7). Proposition 2.E is a rather useful technical result (see, e.g., [63, Problem 6.14]. Proposition 2.F is a synthesis of some sparse results (cf. [28, Proposition I.5.1], [50, Solution 98], [55, Theorem 5.42], [66, Problem 6.37], and Proposition 2.E). For Propositions 2.G and 2.H see [66, Sections 6.3 and 6.4]. The spectral radius formula in Proposition 2.I (see, e.g., [42, p. 22]) ensures that *an operator is uniformly stable if and only if it is similar to a strict contraction* (hint: use the equivalence between (a) and (b) in Corollary 2.11). For Propositions 2.J and 2.K see [66, Sections 6.3 and 6.4]. The examples of spectra in Propositions 2.L, 2.M, and 2.N are widely known (see, e.g., [66, Examples 6.C, 6.D, 6.E, 6.F, and 6.G]). Proposition 2.O deals with the equivalence between uniform and weak stabilities for compact operators, and it is extended to a wider class of operators in Proposition 2.P (see [63, Problems 8.8. and 8.9]). Proposition 2.Q(a) is readily verified, where the invariance of the spectral radius follows by the Gelfand–Beurling formula. Proposition 2.Q(b) is a key result for proving both Theorem 2.8 (the Spectral Mapping Theorem for normal operators) and also Lemma 5.43 of Chapter 5 (for the characterization

of the Browder spectrum in Theorem 5.44) — see, e.g., [76, Theorems 0.3 and 0.4]. For Propositions 2.R, 2.S, and 2.T see [80, Theorem 11.22], [76, Theorem 0.8], and [76, Corollary 0.13], respectively. Proposition 2.T will play a central role in the proof of the Fuglede Theorems of the next chapter (precisely, in Corollary 3.5 and Theorem 3.17).

Suggested Reading

Bachman and Narici [8]

Berberian [17]

Conway [27, 28]

Douglas [34]

Dowson [35]

Dunford and Schwartz [39]

Fillmore [42]

Gustafson and Rao [45]

Halmos [50]

Herrero [55]

Istrǎţescu [58]

Kato [60]

Kubrusly [62, 63, 66]

Radjavi and Rosenthal [76]

Rudin [80]

Taylor and Lay [87]

3

Spectral Theorem

The Spectral Theorem is a milestone in the theory of Hilbert space operators, providing a full statement about the nature and structure of normal operators. For compact normal operators the Spectral Theorem can be completely investigated without requiring any knowledge of measure theory, and this leads to the concept of diagonalization. However, the Spectral Theorem for plain normal operators (the general case) requires some (elementary) measure theory.

3.1 Spectral Theorem for Compact Operators

Throughout this chapter $\mathcal{H}$ will denote a nonzero complex Hilbert space. Recall from Section 1.4 that a bounded weighted sum of projections is an operator T on $\mathcal{H}$ such that

$$Tx = \sum_{\gamma \in \Gamma} \lambda_\gamma E_\gamma x$$

for every $x \in \mathcal{H}$, where $\{E_\gamma\}_{\gamma \in \Gamma}$ is a resolution of the identity on $\mathcal{H}$ made up of nonzero projections (i.e., $\{E_\gamma\}_{\gamma \in \Gamma}$ is an orthogonal family of nonzero orthogonal projections such that $\sum_{\gamma \in \Gamma} E_\gamma x = x$ for every $x \in \mathcal{H}$), and $\{\lambda_\gamma\}_{\gamma \in \Gamma}$ is a bounded family of scalars. *Every bounded weighted sum of projections is normal* (Corollary 1.9). The spectrum of a bounded weighted sum of projections is characterized next as the closure of the set $\{\lambda_\gamma\}_{\gamma \in \Gamma}$,

$$\sigma(T) = \big\{\lambda \in \mathbb{C} \colon \lambda = \lambda_\gamma \text{ for some } \lambda \in \Gamma\big\}^-,$$

so that, since T is normaloid, $\|T\| = r(T) = \sup_{\gamma \in \Gamma} |\lambda_\gamma|$.

Lemma 3.1. *If $T \in \mathcal{B}[\mathcal{H}]$ is a weighted sum of projections, then*

$$\sigma_P(T) = \big\{\lambda \in \mathbb{C} \colon \lambda = \lambda_\gamma \text{ for some } \gamma \in \Gamma\big\}, \qquad \sigma_R(T) = \varnothing, \qquad \text{and}$$

$$\sigma_C(T) = \big\{\lambda \in \mathbb{C} \colon \inf_{\gamma \in \Gamma} |\lambda - \lambda_\gamma| = 0 \text{ and } \lambda \neq \lambda_\gamma \text{ for all } \gamma \in \Gamma\big\}.$$

Proof. Let $T = \sum_{\gamma \in \Gamma} \lambda_\gamma E_\gamma$ be a bounded weighted sum of projections so that $\{E_\gamma\}_{\gamma \in \Gamma}$ is a resolution of the identity on $\mathcal{H}$ and $\{\lambda_\gamma\}_{\gamma \in \Gamma}$ is a bounded sequence of scalars (see Section 1.4). Take an arbitrary x in $\mathcal{H}$. Since $\{E_\gamma\}_{\gamma \in \Gamma}$ is

a resolution of the identity on $\mathcal{H}$, we get $x = \sum_{\gamma \in \Gamma} E_\gamma x$, and the general version of the Pythagorean Theorem (see the Orthogonal Structure Theorem of Section 1.3) ensures that $\|x\|^2 = \sum_{\gamma \in \Gamma} \|E_\gamma x\|^2$. Take any scalar $\lambda \in \mathbb{C}$. Thus $(\lambda I - T)x = \sum_{\gamma \in \Gamma}(\lambda - \lambda_\gamma)E_\gamma x$ and so $\|(\lambda I - T)x\|^2 = \sum_{\gamma \in \Gamma}|\lambda - \lambda_\gamma|^2\|E_\gamma x\|^2$ by the same argument. If $\mathcal{N}(\lambda I - T) \neq \{0\}$, then there exists $x \neq 0$ in $\mathcal{H}$ such that $(\lambda I - T)x = 0$. Hence $\sum_{\gamma \in \Gamma}\|E_\gamma x\|^2 \neq 0$ and $\sum_{\gamma \in \Gamma}|\lambda - \lambda_\gamma|^2\|E_\gamma x\|^2 = 0$, which implies that $\|E_\alpha x\| \neq 0$ for some $\alpha \in \Gamma$ and $|\lambda - \lambda_\alpha|\|E_\alpha x\| = 0$, and therefore $\lambda = \lambda_\alpha$. Conversely, take any $\alpha \in \Gamma$ and any nonzero vector x in $\mathcal{R}(E_\alpha)$ (recall that $\mathcal{R}(E_\gamma) \neq \{0\}$ because $E_\gamma \neq O$ for every $\gamma \in \Gamma$). Since $\mathcal{R}(E_\alpha) \perp \mathcal{R}(E_\gamma)$ whenever $\alpha \neq \gamma$, it follows that $\mathcal{R}(E_\alpha) \perp \bigcup_{\alpha \neq \gamma \in \Gamma} \mathcal{R}(E_\gamma)$, and so $\mathcal{R}(E_\alpha) \subseteq (\bigcup_{\alpha \neq \gamma \in \Gamma} \mathcal{R}(E_\gamma))^\perp = \bigcap_{\alpha \neq \gamma \in \Gamma} \mathcal{R}(E_\gamma)^\perp = \bigcap_{\alpha \neq \gamma \in \Gamma} \mathcal{N}(E_\gamma)$ (see Lemma 1.4 and Theorem 1.8). Thus $x \in \mathcal{N}(E_\gamma)$ for every $\alpha \neq \gamma \in \Gamma$, which implies that $\|(\lambda_\alpha I - T)x\|^2 = \sum_{\gamma \in \Gamma}|\lambda_\alpha - \lambda_\gamma|^2\|E_\gamma x\|^2 = 0$, and therefore $\mathcal{N}(\lambda_\alpha I - T) \neq \{0\}$. Summing up: $\mathcal{N}(\lambda I - T) \neq \{0\}$ if and only if $\lambda = \lambda_\alpha$ for some $\alpha \in \Gamma$. That is,

$$\sigma_P(T) = \big\{\lambda \in \mathbb{C} \colon \lambda = \lambda_\gamma \text{ for some } \gamma \in \Gamma\big\}.$$

Take an arbitrary $\lambda \in \mathbb{C}$ such that $\lambda \neq \lambda_\gamma$ for all $\gamma \in \Gamma$. The weighted sum of projections $\lambda I - T = \sum_{\gamma \in \Gamma}(\lambda - \lambda_\gamma)E_\gamma$ has an inverse, and this inverse is the weighted sum of projections $(\lambda I - T)^{-1} = \sum_{\gamma \in \Gamma}(\lambda - \lambda_\gamma)^{-1}E_\gamma$ because

$$\sum_{\alpha \in \Gamma}(\lambda - \lambda_\alpha)^{-1}E_\alpha \sum_{\beta \in \Gamma}(\lambda - \lambda_\beta)E_\beta x$$
$$= \sum_{\alpha \in \Gamma}\sum_{\beta \in \Gamma}(\lambda - \lambda_\alpha)^{-1}(\lambda - \lambda_\beta)E_\alpha E_\beta x = \sum_{\gamma \in \Gamma}E_\gamma x = x$$

for every x in $\mathcal{H}$ (since the resolution of the identity $\{E_\gamma\}_{\gamma \in \Gamma}$ is an orthogonal family of continuous projections, and the inverse is unique). Moreover, the weighted sum of projections of $(\lambda I - T)^{-1}$ is bounded if and only if $\sup_{\gamma \in \Gamma}|\lambda - \lambda_\gamma|^{-1} < \infty$, that is, if and only if $\inf_{\gamma \in \Gamma}|\lambda - \lambda_\gamma| > 0$. So

$$\rho(T) = \big\{\lambda \in \mathbb{C} \colon \inf_{\gamma \in \Gamma}|\lambda - \lambda_\gamma| > 0\big\}.$$

Since $\sigma_R(T) = \varnothing$ (a weighted sum of projections is normal — Corollary 1.9 and Proposition 2.A(b)), it follows that $\sigma_C(T) = (\mathbb{C}\backslash\rho(T))\backslash\sigma_P(T)$. $\qquad\square$

Compare with Proposition 2.L. Actually, Proposition 2.L is a particular case of Lemma 3.1, since $E_k x = \langle x \, ; e_k \rangle e_k$ for every $x \in \mathcal{H}$ defines the orthogonal projection E_k onto the unidimensional space spanned by e_k.

Lemma 3.2. *A weighted sum of projections* $T \in \mathcal{B}[\mathcal{H}]$ *is compact if and only if the following triple condition holds.*

    (i) $\sigma(T)$ *is countable,*

    (ii) 0 *is the only possible accumulation point of* $\sigma(T)$, *and*

    (iii) $\dim \mathcal{R}(E_\gamma) < \infty$ *for every* γ *such that* $\lambda_\gamma \neq 0$.

Proof. Let $T = \sum_{\gamma \in \Gamma} \lambda_\gamma E_\gamma \in \mathcal{B}[\mathcal{H}]$ be a weighted sum of projections.

Claim. $\mathcal{R}(E_\gamma) \subseteq \mathcal{N}(\lambda_\gamma I - T)$ for every $\gamma \in \Gamma$.

Proof. Take any $\alpha \in \Gamma$. If $x \in \mathcal{R}(E_\alpha)$, then $x = E_\alpha x$ so that $Tx = TE_\alpha x = \sum_{\gamma \in \Gamma} \lambda_\gamma E_\gamma E_\alpha x = \lambda_\alpha E_\alpha x = \lambda_\alpha x$ (since $E_\gamma \perp E_\alpha$ whenever $\alpha \neq \gamma$), and hence $x \in \mathcal{N}(\lambda_\alpha I - T)$, which concludes the proof of the claimed result.

If T is compact, then $\sigma(T)$ is countable and 0 is the only possible accumulation point of $\sigma(T)$ (Corollary 2.20), and $\dim \mathcal{N}(\lambda I - T) < \infty$ whenever $\lambda \neq 0$ (Theorem 1.19) so that $\dim \mathcal{R}(E_\gamma) < \infty$ for every γ such that $\lambda_\gamma \neq 0$ by the above claim. Conversely, if $T = O$, then T is trivially compact. Thus suppose $T \neq O$. Since T is normal (Corollary 1.9), it follows that $r(T) > 0$ (because normal operators are normaloid, i.e., $r(T) = \|T\|$) so that there exists $\lambda \neq 0$ in $\sigma_P(T)$ by Theorem 2.18. If $\sigma(T)$ is countable, then let $\{\lambda_k\}$ be any enumeration of the countable set $\sigma_P(T) \backslash \{0\} = \sigma(T) \backslash \{0\}$. Hence

$$Tx = \sum_k \lambda_k E_k x \ \text{ for every } \ x \in \mathcal{H}$$

(cf. Lemma 3.1), where $\{E_k\}$ is included in a resolution of the identity on $\mathcal{H}$ (which is itself a resolution of the identity on $\mathcal{H}$ if $0 \notin \sigma_P(T)$). If $\{\lambda_k\}$ is finite, say $\{\lambda_k\} = \{\lambda_k\}_{k=1}^n$, then $\mathcal{R}(T) = \sum_{k=1}^n \mathcal{R}(E_k)$. If $\dim \mathcal{R}(E_k) < \infty$ for each k, then $\dim \left(\sum_{k=1}^n \mathcal{R}(E_k) \right)^- < \infty$. Thus T is a finite-rank operator, and so compact (cf. Proposition 1.X). Now suppose $\{\lambda_k\}$ is countably infinite. Since $\sigma(T)$ is a compact set (Theorem 2.1), it follows that the infinite set $\{\lambda_k\}$ has an accumulation point in $\sigma(T)$ — this in fact is the Bolzano–Weierstrass Property that characterizes compact sets in metric spaces. If 0 is the only possible accumulation point of $\sigma(T)$, then 0 is the unique accumulation point of $\{\lambda_k\}$. Therefore, for each integer $n \geq 1$ consider the partition $\{\lambda_k\} = \{\lambda_{k'}\} \cup \{\lambda_{k''}\}$, where $|\lambda_{k'}| \geq \frac{1}{n}$ and $|\lambda_{k''}| < \frac{1}{n}$. Note that $\{\lambda_{k'}\}$ is a finite subset of $\sigma(T)$ (it has no accumulation point), and hence $\{\lambda_{k''}\}$ is an infinite subset of $\sigma(T)$. Set

$$T_n = \sum_{k'} \lambda_{k'} E_{k'} \in \mathcal{B}[\mathcal{H}] \ \text{ for each } \ n \geq 1.$$

We have just seen that $\dim \mathcal{R}(T_n) < \infty$. That is, each T_n is a finite-rank operator. However, since $E_j \perp E_k$ whenever $j \neq k$, we get

$$\|(T - T_n)x\|^2 = \left\| \sum_{k''} \lambda_{k''} E_{k''} x \right\|^2 \leq \sup_{k''} |\lambda_{k''}|^2 \sum_{k''} \|E_{k''} x\|^2 \leq \frac{1}{n^2} \|x\|^2$$

for all $x \in \mathcal{H}$, so that $T_n \xrightarrow{u} T$. Then T is compact by Proposition 1.Z. $\quad\square$

Thus every bounded weighted sum of projections is normal and, if it is compact, then it has a countable set of distinct eigenvalues. The Spectral Theorem for compact operators ensures the converse: *every compact and normal operator T is a (countable) weighted sum of projections, whose weights are precisely the eigenvalues of T.*

Theorem 3.3. (SPECTRAL THEOREM FOR COMPACT OPERATORS). *If an operator $T \in \mathcal{B}[\mathcal{H}]$ on a (nonzero complex) Hilbert space $\mathcal{H}$ is compact and normal, then there exists a unique countable resolution of the identity $\{E_k\}$ on $\mathcal{H}$ and a bounded set of scalars $\{\lambda_k\}$ such that*

$$T = \sum_k \lambda_k E_k,$$

where $\{\lambda_k\} = \sigma_P(T)$, the (nonempty) set of all (distinct) eigenvalues of T, and each E_k is the orthogonal projection onto the eigenspace $\mathcal{N}(\lambda_k I - T)$. If the above countable weighted sum of projections is infinite, then it converges in the (uniform) topology of $\mathcal{B}[\mathcal{H}]$.

Proof. If $T \in \mathcal{B}[\mathcal{H}]$ is compact and normal, then it has a nonempty point spectrum (Corollary 2.21). In fact, the heart of the matter is that T has enough eigenvalues so that its eigenspaces span the Hilbert space $\mathcal{H}$.

Claim. $\left(\sum_{\lambda \in \sigma_P(T)} \mathcal{N}(\lambda I - T) \right)^- = \mathcal{H}.$

Proof. Set $\mathcal{M} = \left(\sum_{\lambda \in \sigma_P(T)} \mathcal{N}(\lambda I - T) \right)^-$, which is a subspace of $\mathcal{H}$. Suppose $\mathcal{M} \neq \mathcal{H}$ so that $\mathcal{M}^\perp \neq \{0\}$. Consider the restriction $T|_{\mathcal{M}^\perp}$ of T to $\mathcal{M}^\perp$. If T is normal, then $\mathcal{M}$ reduces T (Theorem 1.16) so that $\mathcal{M}^\perp$ is T-invariant. Hence $T|_{\mathcal{M}^\perp} \in \mathcal{B}[\mathcal{M}^\perp]$ is normal (Proposition 1.Q). If T is compact, then $T|_{\mathcal{M}^\perp}$ is compact (Proposition 1.V). Thus $T|_{\mathcal{M}^\perp}$ is a compact normal operator on the Hilbert space $\mathcal{M}^\perp \neq \{0\}$, and so $\sigma_P(T|_{\mathcal{M}^\perp}) \neq \varnothing$ by Corollary 2.21. That is, there exist $\lambda \in \mathbb{C}$ and $0 \neq x \in \mathcal{M}^\perp$ such that $T|_{\mathcal{M}^\perp} x = \lambda x$ and so $Tx = \lambda x$. Then $\lambda \in \sigma_P(T)$ and $x \in \mathcal{N}(\lambda I - T) \subseteq \mathcal{M}$. This leads to a contradiction: $0 \neq x \in \mathcal{M} \cap \mathcal{M}^\perp = \{0\}$. Outcome: $\mathcal{M} = \mathcal{H}$.

Since T is compact, the nonempty set $\sigma_P(T)$ is countable (Corollaries 2.20 and 2.21) and bounded (since $T \in \mathcal{B}[\mathcal{H}]$). Then write

$$\sigma_P(T) = \{\lambda_k\},$$

where $\{\lambda_k\}$ is a finite or infinite sequence of distinct elements in $\mathbb{C}$ consisting of all eigenvalues of T. Recall that each $\mathcal{N}(\lambda_k I - T)$ is a subspace of $\mathcal{H}$ (each $\lambda_k I - T$ is bounded). Moreover, since T is normal, Lemma 1.15(a) ensures that $\mathcal{N}(\lambda_k I - T) \perp \mathcal{N}(\lambda_j I - T)$ if $k \neq j$. Thus $\{\mathcal{N}(\lambda_k I - T)\}$ is a sequence of pairwise orthogonal subspaces of $\mathcal{H}$ spanning $\mathcal{H}$; that is, $\mathcal{H} = \left(\sum_k \mathcal{N}(\lambda_k I - T) \right)^-$ by the above claim. Then the sequence $\{E_k\}$ of the orthogonal projections onto each $\mathcal{N}(\lambda_k I - T)$ is a resolution of the identity on $\mathcal{H}$ (see Section 1.4 — recall: E_k onto $\mathcal{N}(\lambda_k I - T)$ is unique). Thus $x = \sum_k E_k x$. Since T is linear and continuous, $Tx = \sum_k T E_k x$ for every $x \in \mathcal{H}$. But $E_k x \in \mathcal{R}(E_k) = \mathcal{N}(\lambda_k I - T)$, and so $T E_k x = \lambda_k E_k x$, for each k and every $x \in \mathcal{H}$. Therefore,

$$Tx = \sum_k \lambda_k E_k x \quad \text{for every } x \in \mathcal{H},$$

and hence T is a countable weighted sum of projections. If it is a finite weighted sum of projections, then we are done. On the other hand, suppose it is an infinite weighted sum of projections. In this case, the above identity says that

$$\sum\nolimits_{k=1}^{n} \lambda_k E_k \xrightarrow{s} T \;\; \text{as} \;\; n \to \infty.$$

That is, the sequence $\{\sum_{k=1}^{n} \lambda_k E_k\}$ converges strongly to T. However, since T is compact, this convergence actually happens in the uniform topology (i.e., $\{\sum_{k=1}^{n} \lambda_k E_k\}$ actually converges uniformly to T). Indeed,

$$\left\| \Big(T - \sum\nolimits_{k=1}^{n} \lambda_k E_k \Big) x \right\|^2 = \left\| \sum\nolimits_{k=n+1}^{\infty} \lambda_k E_k x \right\|^2 = \sum\nolimits_{k=n+1}^{\infty} |\lambda_k|^2 \|E_k x\|^2$$

$$\le \sup_{k \ge n+1} |\lambda_k|^2 \sum\nolimits_{k=n+1}^{\infty} \|E_k x\|^2 \le \sup_{k \ge n} |\lambda_k|^2 \|x\|^2$$

for every integer $n \ge 1$ (reason: $\mathcal{R}(E_j) \perp \mathcal{R}(E_k)$ whenever $j \ne k$, and $x = \sum_{k=1}^{\infty} E_k x$ so that $\|x\|^2 = \sum_{k=1}^{\infty} \|E_k x\|^2$ — see Sections 1.3 and 1.4). Hence

$$0 \le \left\| T - \sum\nolimits_{k=1}^{n} \lambda_k E_k \right\| = \sup_{\|x\|=1} \left\| \Big(T - \sum\nolimits_{k=1}^{n} \lambda_k E_k \Big) x \right\| \le \sup_{k \ge n} |\lambda_k|$$

for all $n \in \mathbb{N}$. Since T is compact and $\{\lambda_n\}$ is a sequence of distinct elements in $\sigma(T)$, it follows that $\lambda_n \to 0$ (cf. Claim in the proof of Corollary 2.20). Thus $\lim_n \sup_{k \ge n} |\lambda_k| = \limsup_n |\lambda_n| = 0$, and so

$$\sum\nolimits_{k=1}^{n} \lambda_k E_k \xrightarrow{u} T \;\; \text{as} \;\; n \to \infty. \qquad \square$$

3.2 Diagonalizable Operators

What the Spectral Theorem for compact (normal) operators says is that, if $T \in \mathcal{B}[\mathcal{H}]$ is compact and normal, then the family of orthogonal projections $\{E_\lambda\}_{\lambda \in \sigma_P(T)}$ onto each eigenspace $\mathcal{N}(\lambda I - T)$ is a countable resolution of the identity on $\mathcal{H}$, and T is a weighted sum of projections. Thus we write

$$T = \sum\nolimits_{\lambda \in \sigma_P(T)} \lambda E_\lambda,$$

which is a uniform limit, and so it implies pointwise convergence (i.e., $Tx = \sum_{\lambda \in \sigma_P(T)} \lambda E_\lambda x$ for every $x \in \mathcal{H}$). This is naturally identified with (i.e., it is unitarily equivalent to) an orthogonal direct sum of scalar operators,

$$T \cong \bigoplus\nolimits_{\lambda \in \sigma_P(T)} \lambda I_\lambda,$$

with each I_λ denoting the identity on each eigenspace $\mathcal{N}(\lambda I - T)$. In fact, $I_\lambda = E_\lambda|_{\mathcal{R}(E_\lambda)}$, where $\mathcal{R}(E_\lambda) = \mathcal{N}(\lambda I - T)$. Thus, under such an identification, it is usual to write

$$T = \bigoplus\nolimits_{\lambda \in \sigma_P(T)} \lambda E_\lambda.$$

These representations are referred to as the *spectral decomposition* of a compact normal operator T. The Spectral Theorem for compact (normal) operators can be restated in terms of an orthonormal basis for $\mathcal{N}(T)^\perp$ consisting of eigenvectors of T, as follows.

Corollary 3.4. *Let $T \in \mathcal{B}[\mathcal{H}]$ be compact and normal.*

(a) *For each $\lambda \in \sigma_P(T)\backslash\{0\}$ there is a finite orthonormal basis $\{e_k(\lambda)\}_{k=1}^{n_\lambda}$ for $\mathcal{N}(\lambda I - T)$ consisting entirely of eigenvectors of T.*

(b) *The set $\{e_k\} = \bigcup_{\lambda \in \sigma_P(T)\backslash\{0\}} \{e_k(\lambda)\}_{k=1}^{n_\lambda}$ is a countable orthonormal basis for $\mathcal{N}(T)^\perp$ made up of eigenvectors of T.*

(c) $Tx = \sum_{\lambda \in \sigma_P(T)\backslash\{0\}} \lambda \sum_{k=1}^{n_\lambda} \langle x \,; e_k(\lambda) \rangle e_k(\lambda)$ *for every $x \in \mathcal{H}$.*

(d) $Tx = \sum_k \nu_k \langle x \,; e_k \rangle e_k$ *for every $x \in \mathcal{H}$, where $\{\nu_k\}$ is a sequence containing all nonzero eigenvalues of T including multiplicity (i.e., finitely repeated according to the dimension of the respective eigenspace).*

Proof. We have already seen (in the proof of the previous theorem) that $\sigma_P(T)$ is nonempty and countable. Recall that $\sigma_P(T) = \{0\}$ if and only if $T = O$ (since T is normaloid; see Corollary 2.21) or, equivalently, if and only if $\mathcal{N}(T)^\perp = \{0\}$ (i.e., $\mathcal{N}(T) = \mathcal{H}$). If $T = O$, then the above assertions hold trivially $(\sigma_P(T)\backslash\{0\} = \varnothing$, $\{e_k\} = \varnothing$, $\mathcal{N}(T)^\perp = \{0\}$ and $Tx = 0x = 0$ for every $x \in \mathcal{H}$ because the empty sum is null). Thus suppose $T \neq O$ (so that $\mathcal{N}(T)^\perp \neq \{0\}$), and take an arbitrary $\lambda \neq 0$ in $\sigma_P(T)$. Theorem 1.19 ensures that $\dim \mathcal{N}(\lambda I - T)$ is finite, say, $\dim \mathcal{N}(\lambda I - T) = n_\lambda$ for some positive integer n_λ. This ensures the existence of a *finite* orthonormal basis $\{e_k(\lambda)\}_{k=1}^{n_\lambda}$ for the Hilbert space $\mathcal{N}(\lambda I - T) \neq \{0\}$. Recall that each $e_k(\lambda)$ is an eigenvector of T (since $0 \neq e_k(\lambda) \in \mathcal{N}(\lambda I - T)$).

Claim. $\bigcup_{\lambda \in \sigma_P(T)\backslash\{0\}} \{e_k(\lambda)\}_{k=1}^{n_\lambda}$ is a orthonormal basis for $\mathcal{N}(T)^\perp$.

Proof. Theorem 3.3 ensures that

$$\mathcal{H} = \left(\sum_{\lambda \in \sigma_P(T)} \mathcal{N}(\lambda I - T) \right)^-.$$

Since $\{\mathcal{N}(\lambda I - T)\}_{\lambda \in \sigma_P(T)}$ is a nonempty family of orthogonal subspaces of $\mathcal{H}$ (by Lemma 1.15(a)), it follows that

$$\mathcal{N}(T) = \bigcap_{\lambda \in \sigma_P(T)\backslash\{0\}} \mathcal{N}(\lambda I - T)^\perp = \left(\sum_{\lambda \in \sigma_P(T)\backslash\{0\}} \mathcal{N}(\lambda I - T) \right)^\perp$$

(see, e.g., [66, Problem 5.8(b,e)]). Hence (see Section 1.3),

$$\mathcal{N}(T)^\perp = \left(\sum_{\lambda \in \sigma_P(T)\backslash\{0\}} \mathcal{N}(\lambda I - T) \right)^-.$$

Therefore, since $\{\mathcal{N}(\lambda I - T)\}_{\lambda \in \sigma_P(T)}$ is a family of orthogonal subspaces, the claimed result follows by part (a).

Note that the set $\{e_k\} = \bigcup_{\lambda \in \sigma_P(T)\backslash\{0\}} \{e_k(\lambda)\}_{k=1}^{n_\lambda}$ is countable (a countable union of countable sets is countable). Finally, consider the decomposition $\mathcal{H} = \mathcal{N}(T) + \mathcal{N}(T)^\perp$ (see Section 1.3), and take an arbitrary $x \in \mathcal{H}$ so that

$x = u + v$ with $u \in \mathcal{N}(T)$ and $v \in \mathcal{N}(T)^\perp$. Consider the Fourier series expansion $v = \sum_k \langle v\,;e_k\rangle e_k = \sum_{\lambda \in \sigma_P(T)\setminus\{0\}} \sum_{k=1}^{n_\lambda} \langle v\,;e_k(\lambda)\rangle e_k(\lambda)$ of v in terms of the orthonormal basis $\{e_k\} = \bigcup_{\lambda \in \sigma_P(T)\setminus\{0\}} \{e_k(\lambda)\}_{k=1}^{n_\lambda}$ for the Hilbert space $\mathcal{N}(T)^\perp \neq \{0\}$. Since T is linear and continuous, and since $Te_k(\lambda) = \lambda e_k(\lambda)$ for each $k = 1,\dots,n_\lambda$ and each $\lambda \in \sigma_P(T)\setminus\{0\}$, it follows that

$$
\begin{aligned}
Tx \;=\; Tu + Tv \;&=\; Tv \\
&= \sum_{\lambda \in \sigma_P(T)\setminus\{0\}} \sum_{k=1}^{n_\lambda} \langle v\,;e_k(\lambda)\rangle \, Te_k(\lambda) \\
&= \sum_{\lambda \in \sigma_P(T)\setminus\{0\}} \lambda \sum_{k=1}^{n_\lambda} \langle v\,;e_k(\lambda)\rangle \, e_k(\lambda).
\end{aligned}
$$

However, $\langle x\,;e_k(\lambda)\rangle = \langle u\,;e_k(\lambda)\rangle + \langle v\,;e_k(\lambda)\rangle = \langle v\,;e_k(\lambda)\rangle$ because $u \in \mathcal{N}(T)$ and $e_k(\lambda) \in \mathcal{N}(T)^\perp$. $\square$

Remark. Observe that *if $T \in \mathcal{B}[\mathcal{H}]$ is compact and normal, and if $\mathcal{H}$ is nonseparable, then $0 \in \sigma_P(T)$ and $\mathcal{N}(T)$ is nonseparable.* Indeed, if $T \neq O$ (otherwise the above statement is trivial), then $\mathcal{N}(T)^\perp \neq \{0\}$ is separable (i.e., it has a countable orthonormal basis) by Corollary 3.4 and so $\mathcal{N}(T)$ is nonseparable whenever $\mathcal{H} = \mathcal{N}(T) + \mathcal{N}(T)^\perp$ is nonseparable. Moreover, if $0 \notin \sigma_P(T)$, then $\mathcal{N}(T) = \{0\}$, and therefore $\mathcal{H} = \mathcal{N}(T)^\perp$ is separable.

The expression of a compact normal operator T in Corollary 3.4(c) (as well as the very spectral decomposition of it) says that T is a diagonalizable operator in the sense that $T = T|_{\mathcal{N}(T)^\perp} \oplus O$ (recall that $\mathcal{N}(T)$ reduces T) and $T|_{\mathcal{N}(T)^\perp} \in \mathcal{B}[\mathcal{N}(T)^\perp]$ is a diagonal operator with respect to a countable orthonormal basis $\{e_k\}$ for the separable Hilbert space $\mathcal{N}(T)^\perp$.

Generalizing: An operator $T \in \mathcal{B}[\mathcal{H}]$ (not necessarily compact) acting on any Hilbert space $\mathcal{H}$ (not necessarily separable) is *diagonalizable* if there exist a resolution of the identity $\{E_\gamma\}_{\gamma \in \Gamma}$ on $\mathcal{H}$ and a bounded family of scalars $\{\lambda_\gamma\}_{\gamma \in \Gamma}$ such that

$$
Tu = \lambda_\gamma u \quad \text{whenever} \quad u \in \mathcal{R}(E_\gamma).
$$

Take an arbitrary $x = \sum_{\gamma \in \Gamma} E_\gamma x$ in $\mathcal{H}$. Since T is linear and continuous, it follows that $Tx = \sum_{\gamma \in \Gamma} TE_\gamma x = \sum_{\gamma \in \Gamma} \lambda_\gamma E_\gamma x$, and hence T is a weighted sum of projections (which is normal). Thus we write

$$
T = \sum_{\gamma \in \Gamma} \lambda_\gamma E_\gamma \quad \text{or} \quad T = \bigoplus_{\gamma \in \Gamma} \lambda_\gamma E_\gamma.
$$

Conversely, if T is a weighted sum of projections (i.e., $Tx = \sum_{\gamma \in \Gamma} \lambda_\gamma E_\gamma x$ for every $x \in \mathcal{H}$), then $Tu = \sum_{\gamma \in \Gamma} \lambda_\gamma E_\gamma u = \sum_{\gamma \in \Gamma} \lambda_\gamma E_\gamma E_\alpha u = \lambda_\alpha u$ for every $u \in \mathcal{R}(E_\alpha)$ (since $E_\gamma E_\alpha = O$ whenever $\gamma \neq \alpha$ and $u = E_\alpha u$ whenever $u \in \mathcal{R}(E_\alpha)$), and so T is diagonalizable. Conclusion: *An operator T on $\mathcal{H}$ is diagonalizable if and only if it is a weighted sum of projections for some bounded sequence of scalars $\{\lambda_\gamma\}_{\gamma \in \Gamma}$ and some resolution of the identity $\{E_\gamma\}_{\gamma \in \Gamma}$ on*

$\mathcal{H}$. In this case, $\{E_\gamma\}_{\gamma \in \Gamma}$ is said to *diagonalize T*. (See also Proposition 3.A — just recall that the identity on any infinite-dimensional Hilbert space $\mathcal{H}$ is trivially diagonalizable (a diagonal, actually, if $\mathcal{H} = \ell^2_\Gamma$), and thus normal, but not compact — cf. Proposition 1.Y.)

The implication (b) $\Rightarrow$ (d) in Corollary 3.5 is the compact version of the Fuglede Theorem (Theorem 3.17) that will be developed in Section 3.5.

Corollary 3.5. *Suppose T is a compact operator on a Hilbert space $\mathcal{H}$.*

(a) *T is normal if and only if it is diagonalizable.*

Let $\{E_k\}$ be a resolution of the identity on $\mathcal{H}$ that diagonalizes a compact and normal operator $T \in \mathcal{B}[\mathcal{H}]$ into its spectral decomposition, and take any operator $S \in \mathcal{B}[\mathcal{H}]$. The following assertions are pairwise equivalent.

(b) *S commutes with T.*

(c) *S commutes with T^*.*

(d) *S commutes with every E_k.*

(e) *$\mathcal{R}(E_k)$ reduces S for every k.*

Proof. Take a compact operator T on $\mathcal{H}$. If T is normal, then the Spectral Theorem ensures that it is diagonalizable. The converse is trivial since a diagonalizable operator is normal. This proves (a). From now on suppose T is compact and normal, and consider its spectral decomposition,

$$T = \sum_k \lambda_k E_k$$

as in Theorem 3.3, where $\{E_k\}$ is a resolution of the identity on $\mathcal{H}$ and $\{\lambda_k\} = \sigma_P(T)$ is the set of all (distinct) eigenvalues of T. Recall that

$$T^* = \sum_k \overline{\lambda}_k E_k.$$

The central result in the above statement is the implication (b) $\Rightarrow$ (e). The main idea behind its proof relies on Proposition 2.T (also see, e.g., [76, Theorem 1.16]). Take any operator S on $\mathcal{H}$. Suppose $ST = TS$. Since

$$E_k T = \lambda_k E_k = T E_k$$

(recall: $E_k T = \sum_j \lambda_j E_k E_j = \lambda_k E_k = \sum_j \lambda_j E_j E_k = T E_k$), it follows that each $\mathcal{R}(E_k) = \mathcal{N}(\lambda_k I - T)$ reduces T (Proposition 1.I) so that

$$E_k T|_{\mathcal{R}(E_k)} = T E_k|_{\mathcal{R}(E_k)} = T|_{\mathcal{R}(E_k)} = \lambda_k I : \mathcal{R}(E_k) \to \mathcal{R}(E_k),$$

and hence, on each $\mathcal{R}(E_k)$,

$$\begin{aligned}(E_j S E_k) T|_{\mathcal{R}(E_k)} &= (E_j S E_k)(T E_k) = E_j S T E_k = E_j T S E_k \\ &= (T E_j)(E_j S E_k) = T|_{\mathcal{R}(E_j)}(E_j S E_k)\end{aligned}$$

whenever $j \neq k$. Moreover, since $\sigma(T|_{\mathcal{R}(E_k)}) = \{\lambda_k\}$ and $\lambda_k \neq \lambda_j$, we get

$$\sigma(TE_k) \cap \sigma(TE_j) = \varnothing$$

and so, according to Proposition 2.T, for $j \neq k$,

$$E_j SE_k = O.$$

Thus (as each E_k is self-adjoint and $\{E_k\}$ is a resolution of the identity)

$$\langle SE_k x \,; y \rangle = \Big\langle SE_k x \,; \sum_j E_j y \Big\rangle = \sum_j \langle E_j SE_k x \,; y \rangle = \langle E_k SE_k x \,; y \rangle$$

for every $x, y \in \mathcal{H}$. Then

$$SE_k = E_k SE_k,$$

which means that $\mathcal{R}(E_k)$ is S-invariant (Proposition 1.I) for every k. Hence $\mathcal{R}(E_k)^\perp = \big(\sum_{j \neq k} \mathcal{R}(E_j)\big)^- = \bigvee_{j \neq k} \mathcal{R}(E_j)$ (because $\{E_k\}$ is a resolution of the identity on $\mathcal{H}$). Thus, since $\mathcal{R}(E_j)$ is S-invariant for every j, it follows that $\mathcal{R}(E_k)^\perp$ also is S-invariant. Therefore,

$$\mathcal{R}(E_k) \text{ reduces } S \text{ for every } k.$$

This proves that (b) implies (e). But (e) is equivalent to (d) by Proposition 1.I. Note that (d) implies (c). Indeed, if $SE_k = E_k S$ for every k, then $ST^* = \sum_k \overline{\lambda}_k SE_k = \sum_k \overline{\lambda}_k E_k S = T^* S$ (as S is linear and continuous). Thus (b) implies (c), and therefore (c) implies (b) because $T^{**} = T$. $\qquad\square$

3.3 Spectral Measure

If T is a compact operator, then its point spectrum is nonempty and countable, which may not hold for noncompact (normal) operators. But this is not the main role played by compact operators in the Spectral Theorem — we can deal with an uncountable weighted sum of projections $Tx = \sum_{\gamma \in \Gamma} \lambda_\gamma E_\gamma x$. What is actually special with a compact operator is that a compact normal operator not only has a nonempty point spectrum but it has enough eigenspaces to span $\mathcal{H}$; that is, $\big(\sum_{\lambda \in \sigma_P(T)} \mathcal{N}(\lambda I - T)\big)^- = \mathcal{H}$ (see proof of Theorem 3.3). That makes the difference, since a normal (noncompact) operator may have an empty point spectrum or it may have eigenspaces but not enough to span the whole space $\mathcal{H}$. However, the Spectral Theorem survives the lack of compactness if the point spectrum is replaced with the whole spectrum (which is never empty). Such an approach for the general case of the Spectral Theorem (i.e., for normal, not necessarily compact operators) requires measure theory.

Let $\overline{\mathbb{R}}$ stand for extended real line. Take a measure space $(\Omega, \mathcal{A}_\Omega, \mu)$, where $\mathcal{A}_\Omega$ is a σ-algebra of subsets of a nonempty set Ω and $\mu \colon \mathcal{A}_\Omega \to \overline{\mathbb{R}}$ is a nonnegative measure on $\mathcal{A}_\Omega$, which means that $\mu(\Lambda) \geq 0$ for every $\Lambda \in \mathcal{A}_\Omega$. The sets in A_Ω are called *measurable* sets (or $\mathcal{A}_\Omega$-measurable). Observe that $\mathcal{A}_\Omega \neq \{\varnothing\}$

since $\Omega \neq \varnothing$. The measure μ is said to be *positive* if it is nonnegative and not null (i.e., $\mu(\Omega) > 0$). It is *finite* if $\mu(\Omega) < \infty$ (i.e., $\mu \colon \mathcal{A}_\Omega \to \mathbb{R}$), and *$\sigma$-finite* if Ω is a countable union of measurable sets of finite measure. Let $\mathbb{F}$ denote either the real line $\mathbb{R}$ or the complex plane $\mathbb{C}$. A *Borel σ-algebra* of subsets of $\mathbb{F}$ (or a σ-algebra of *Borel sets*) is the σ-algebra $\mathcal{A}_\mathbb{F}$ generated by the usual (metric) topology of $\mathbb{F}$. (Note that for the σ-algebra $\mathcal{A}_\mathbb{R}$ of Borel subsets of $\mathbb{R}$, this coincides with the standard Borel σ-algebra generated by the open intervals.) It is clear that all open, and so all closed, subsets of $\mathbb{F}$ are Borel sets (i.e., all open or closed subsets of $\mathbb{F}$ lie in $\mathcal{A}_\mathbb{F}$). From now on suppose the nonempty Ω is a Borel subset of $\mathbb{F}$ (i.e., a set in the Borel σ-algebra $\mathcal{A}_\mathbb{F}$) and set $\mathcal{A}_\Omega = \wp(\Omega) \cap \mathcal{A}_\mathbb{F}$, where $\wp(\Omega)$ is the *power set* of Ω (the collection of all subsets of Ω). This $\mathcal{A}_\Omega$ is a σ-algebra, referred to as the σ-algebra *of Borel subsets of Ω*. A measure μ on $\mathcal{A}_\Omega$ is said to be a *Borel measure* if $\mu(K) < \infty$ for every compact set $K \subseteq \Omega$ (which is in $\mathcal{A}_\Omega$). Every Borel measure is σ-finite, and every finite measure is a Borel measure. The *support* of a measure μ on $\mathcal{A}_\Omega$ is the set support$(\mu) = \Omega \backslash \bigcup \{ \Lambda \in \mathcal{A}_\Omega \colon \Lambda$ is open and $\mu(\Lambda) = 0 \}$, which is a closed set such that its complement is the largest open set of measure zero. Equivalently, support(μ) *is the smallest closed set whose complement has measure zero*. In the vast literature on measure theory, the reader is referred, for instance, to [11], [21], [46], [65], [78], or [79] (which simply reflects the author's preference but, by no means, exhausts any list).

Remark. A set $K \subseteq \Omega \subseteq \mathbb{F}$ is compact if it is complete and totally bounded (in $\mathbb{F}$, total boundedness coincides with boundedness). So $K \subseteq \Omega$ is compact in the relative topology of Ω if and only if it is in $\mathbb{F}$ (where compact means closed and bounded). But *open* and *closed* in the preceding paragraph refer to the topology of Ω. Note: if Ω is closed in $\mathbb{F}$, then *closed* in Ω and in $\mathbb{F}$ coincide.

Definition 3.6. Let Ω be a nonempty Borel subset of $\mathbb{C}$ and let $\mathcal{A}_\Omega$ be the σ-algebra of Borel subsets of Ω. A (complex) *spectral measure* in a (complex) Hilbert space $\mathcal{H}$ is a mapping $E \colon \mathcal{A}_\Omega \to \mathcal{B}[\mathcal{H}]$ such that

(a) $E(\Lambda)$ is an orthogonal projection for every $\Lambda \in \mathcal{A}_\Omega$,

(b) $E(\varnothing) = O$ and $E(\Omega) = I$,

(c) $E(\Lambda_1 \cap \Lambda_2) = E(\Lambda_1)E(\Lambda_2)$ for every $\Lambda_1, \Lambda_2 \in \mathcal{A}_\Omega$,

(d) $E\left(\bigcup_k \Lambda_k \right) = \sum_k E(\Lambda_k)$ whenever $\{\Lambda_k\}$ is a countable collection of pairwise disjoint sets in $\mathcal{A}_\Omega$ (i.e., E is countably additive).

If $\{\Lambda_k\}$ is a countably infinite collection of pairwise disjoint sets in $\mathcal{A}_\Omega$, then the identity in (d) means convergence in the strong topology:

$$\sum_{k=1}^{n} E(\Lambda_k) \xrightarrow{\ s\ } E\left(\bigcup_k \Lambda_k \right).$$

Indeed, since $\Lambda_j \cap \Lambda_k = \varnothing$ if $j \neq k$, it follows by properties (b) and (c) that $E(\Lambda_j)E(\Lambda_k) = E(\Lambda_j \cap \Lambda_k) = E(\varnothing) = O$ for $j \neq k$, so that $\{E(\Lambda_k)\}$ is an

orthogonal sequence of orthogonal projections in $\mathcal{B}[\mathcal{H}]$. Then, according to Proposition 1.K, $\{\sum_{k=1}^{n} E(\Lambda_k)\}$ converges strongly to the orthogonal projection in $\mathcal{B}[\mathcal{H}]$ onto $\left(\sum_k \mathcal{R}(E(\Lambda_k))\right)^{-} = \bigvee \left(\bigcup_k \mathcal{R}(E(\Lambda_k))\right)$. Thus what property (d) says is that $E\left(\bigcup_k \Lambda_k\right)$ coincides with the orthogonal projection in $\mathcal{B}[\mathcal{H}]$ onto $\bigvee \left(\bigcup_k \mathcal{R}(E(\Lambda_k))\right)$. This generalizes the concept of a resolution of the identity on $\mathcal{H}$. In fact, if $\{\Lambda_k\}$ is a partition of Ω, then the orthogonal sequence of orthogonal projections $\{E(\Lambda_k)\}$ is such that

$$\sum_{k=1}^{n} E(\Lambda_k) \xrightarrow{s} E\left(\bigcup_k \Lambda_k\right) = E(\Omega) = I.$$

Let $E\colon \mathcal{A}_\Omega \to \mathcal{B}[\mathcal{H}]$ be any spectral measure into $\mathcal{B}[\mathcal{H}]$. For each pair of vectors $x, y \in \mathcal{H}$ consider the mapping $\pi_{x,y}\colon \mathcal{A}_\Omega \to \mathbb{C}$ defined by

$$\pi_{x,y}(\Lambda) = \langle E(\Lambda)x\,;y\rangle$$

for every $\Lambda \in \mathcal{A}_\Omega$. The mapping $\pi_{x,y}$ is an ordinary complex-valued (countably additive) measure on $\mathcal{A}_\Omega$. For each $x, y \in \mathcal{H}$ consider the measure space $(\Omega, \mathcal{A}_\Omega, \pi_{x,y})$. Let $B(\Omega)$ denote the algebra of all complex-valued *bounded* $\mathcal{A}_\Omega$-*measurable* functions $\phi\colon \Omega \to \mathbb{C}$ on Ω. The integral of a function ϕ in $B(\Omega)$ with respect to the measure $\pi_{x,y}$, viz., $\int \phi\, d\pi_{x,y}$, will also be denoted by $\int \phi(\lambda)\, d\pi_{x,y}$, or by $\int \phi\, d\langle E_\lambda x\,;y\rangle$, or by $\int \phi(\lambda)\, d\langle E_\lambda x\,;y\rangle$.

Lemma 3.7. *Let* $E\colon \mathcal{A}_\Omega \to \mathcal{B}[\mathcal{H}]$ *be a spectral measure. For every bounded* $\mathcal{A}_\Omega$-*measurable function* $\phi \in B(\Omega)$ *there is a unique* $F \in \mathcal{B}[\mathcal{H}]$ *such that*

$$\langle Fx\,;y\rangle = \int \phi(\lambda)\, d\langle E_\lambda x\,;y\rangle \ \ \text{for every} \ \ x,y \in \mathcal{H}.$$

Proof. Let $\phi\colon \Omega \to \mathbb{C}$ be a bounded function and set $\|\phi\|_\infty = \sup_{\lambda \in \Omega} |\phi(\lambda)|$. Define the sesquilinear form $f\colon \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ by

$$f(x,y) = \int \phi(\lambda)\, d\langle E_\lambda x\,;y\rangle$$

for every $x, y \in \mathcal{H}$. Since $\langle E(\cdot)x\,;x\rangle$ is a positive measure, for every $x \in \mathcal{H}$,

$$|f(x,x)| \le \|\phi\|_\infty \int d\langle E_\lambda x\,;x\rangle = \|\phi\|_\infty \langle E(\Omega)x\,;x\rangle = \|\phi\|_\infty \|x\|^2.$$

Thus, by the polarization identity for sesquilinear forms (Proposition 1.A(a_1)), and by the parallelogram law (Proposition 1.A(b)), we get

$$|f(x,y)| \le \|\phi\|_\infty \left(\|x\|^2 + \|y\|^2\right)$$

for every $x, y \in \mathcal{H}$. Then f is bounded ($\sup_{\|x\|=\|y\|=1} |f(x,y)| < \infty$) and so is the linear functional $f(\cdot, y)\colon \mathcal{H} \to \mathbb{C}$ for each $y \in \mathcal{H}$. Thus, by the Riesz Representation Theorem in Hilbert space (see, e.g., [66, Theorem 5.62]), for each $y \in \mathcal{H}$ there is a unique $z_y \in \mathcal{H}$ such that $f(x,y) = \langle x\,;z_y\rangle$ for each

$x \in \mathcal{H}$. This establishes a mapping $F^{\#} \colon \mathcal{H} \to \mathcal{H}$ that assigns to each $y \in \mathcal{H}$ a unique $z_y \in \mathcal{H}$ such that $f(x,y) = \langle x \,;\, F^{\#} y \rangle$ for every $x, y \in \mathcal{H}$. It is easy to show that $F^{\#}$ is unique and lies in $\mathcal{B}[\mathcal{H}]$. Hence $F = (F^{\#})^*$ is the unique operator in $\mathcal{B}[\mathcal{H}]$ such that $\langle Fx \,;\, y \rangle = f(x,y)$ for every $x, y \in \mathcal{H}$. $\square$

Notation. The unique $F \in \mathcal{B}[\mathcal{H}]$ such that $\langle Fx \,;\, y \rangle = \int \phi(\lambda) \, d\langle E_\lambda x \,;\, y \rangle$ for every $x, y \in \mathcal{H}$ as in Lemma 3.7 is usually denoted by

$$F = \int \phi(\lambda) \, dE_\lambda.$$

In particular, if ϕ is the characteristic function χ_Λ of a set $\Lambda \in \mathcal{A}_\Omega$, then $\langle E(\Lambda)x \,;\, y \rangle = \pi_{x,y}(\Lambda) = \int_\Lambda d\pi_{x,y} = \int \chi_\Lambda \, d\pi_{x,y} = \int \chi_\Lambda(\lambda) \, d\langle E_\lambda x \,;\, y \rangle$ for every $x, y \in \mathcal{H}$, and so the orthogonal projection $E(\Lambda) \in \mathcal{B}[\mathcal{H}]$ is denoted by

$$E(\Lambda) = \int \chi_\Lambda(\lambda) \, dE_\lambda = \int_\Lambda dE_\lambda.$$

Lemma 3.8. *Let $E \colon \mathcal{A}_\Omega \to \mathcal{B}[\mathcal{H}]$ be a spectral measure. Suppose ϕ and ψ are (bounded $\mathcal{A}_\Omega$-measurable) functions in $B(\Omega)$, and consider the operators $F = \int \phi(\lambda) \, dE_\lambda$ and $G = \int \psi(\lambda) \, dE_\lambda$ in $\mathcal{B}[\mathcal{H}]$. Then*

(a) $\quad F^* = \int \overline{\phi(\lambda)} \, dE_\lambda \quad$ and $\quad$ (b) $\quad FG = \int \phi(\lambda)\, \psi(\lambda) \, dE_\lambda.$

Proof. Given ϕ and ψ in $B(\Omega)$, consider the operators $F = \int \phi(\lambda) \, dE_\lambda$ and $G = \int \psi(\lambda) \, dE_\lambda$ in $\mathcal{B}[\mathcal{H}]$ (cf. Lemma 3.7). Take an arbitrary pair $x, y \in \mathcal{H}$.

(a) Observe from the proof of Lemma 3.7 that

$$\langle F^* x \,;\, y \rangle = \overline{\langle y \,;\, F^* x \rangle} = \overline{f(y,x)} = \int \overline{\phi(\lambda) \, d\langle E_\lambda y \,;\, x \rangle} = \int \overline{\phi(\lambda)} \, d\langle E_\lambda x \,;\, y \rangle.$$

(b) Set $\pi = \pi_{x,y}$. Since $\langle Fx \,;\, y \rangle = \int \phi \, d\pi = \int \phi(\lambda) \, d\langle E_\lambda x \,;\, y \rangle$, it follows that

$$\langle FGx \,;\, y \rangle = \int \phi(\lambda) \, d\langle E_\lambda Gx \,;\, y \rangle.$$

Moreover, since $\langle Gx \,;\, y \rangle = \int \psi \, d\pi = \int \psi(\lambda) \, d\langle E_\lambda x \,;\, y \rangle$, it also follows that

$$\langle E(\Lambda)Gx \,;\, y \rangle = \langle Gx \,;\, E(\Lambda)y \rangle = \int \psi(\lambda) \, d\langle E_\lambda x \,;\, E(\Lambda)y \rangle$$

for every $\Lambda \in \mathcal{A}_\Omega$. Therefore,

$$\langle FGx \,;\, y \rangle = \int \phi \, d\nu,$$

where $\nu = \nu_{x,y} \colon \mathcal{A}_\Omega \to \mathbb{C}$ is the measure defined, for each $\Lambda \in \mathcal{A}_\Omega$, by

$$\nu(\Lambda) = \int \psi(\lambda) \, d\langle E_\lambda x \,;\, E(\Lambda)y \rangle.$$

Thus

$$\int_\Lambda d\nu = \nu(\Lambda) = \int \psi(\lambda)\, d\langle E(\Lambda)E_\lambda x\,;y\rangle = \int_\Lambda \psi(\lambda)\, d\langle E_\lambda x\,;y\rangle = \int_\Lambda \psi\, d\pi$$

for each $\Lambda \in \mathcal{A}_\Omega$. This is usually written as $d\nu = \psi\, d\pi$, meaning that $\psi = \frac{d\nu}{d\pi}$ is the Radon–Nikodým derivative of π with respect to ν. In fact, $\int_\Lambda d\nu = \int_\Lambda \psi\, d\pi$ for every $\Lambda \in \mathcal{A}_\Omega$ implies that $\int \phi\, d\nu = \int \phi\,\psi\, d\pi$ for every $\phi \in B(\Omega)$. Hence

$$\langle FGx\,;y\rangle = \int \phi\, d\nu = \int \phi\,\psi\, d\pi = \int \phi(\lambda)\,\psi(\lambda)\, d\langle E_\lambda x\,;y\rangle. \qquad \square$$

Lemma 3.9. *Given a spectral measure $E\colon \mathcal{A}_\Omega \to \mathcal{B}[\mathcal{H}]$ and a bounded $\mathcal{A}_\Omega$-measurable function $\phi \in B(\Omega)$, the operator $F = \int \phi(\lambda)\, dE_\lambda$ is normal.*

Proof. According to Lemmas 3.7 and 3.8,

$$F^*F = \int |\phi(\lambda)|^2\, dE_\lambda = FF^*. \qquad \square$$

Lemma 3.10. *Let $E\colon \mathcal{A}_\Omega \to \mathcal{B}[\mathcal{H}]$ be a spectral measure, where Ω is a compact subset of $\mathbb{C}$. The operator $F = \int \lambda\, dE_\lambda$ is well defined in $\mathcal{B}[\mathcal{H}]$, is normal, and*

$$p(F,F^*) = \int p(\lambda, \overline{\lambda})\, dE_\lambda = p(F^*, F)$$

for every polynomial $p(\cdot,\cdot)\colon \Omega \times \Omega \to \mathbb{C}$ in λ and $\overline{\lambda}$.

Proof. Since $\Omega \neq \varnothing$ is compact, the identity function on Ω is bounded and $\mathcal{A}_\Omega$-measurable, and so F is well defined and normal by Lemmas 3.7 and 3.9. If $p(\lambda, \overline{\lambda}) = \sum_{i,j=0}^{n,m} \alpha_{i,j} \lambda^i \overline{\lambda}^j$, then set $p(F,F^*) = \sum_{i,j=0}^{n,m} \alpha_{i,j} F^i F^{*j} = p(F^*, F)$. Thus (by linearity of the integral) the result follows by Lemma 3.8 (since $p(\cdot,\overline{\cdot})\colon \Omega \to \mathbb{C}$ such that $\lambda \mapsto p(\lambda, \overline{\lambda})$ lies in $B(\Omega)$ because Ω is bounded). $\quad\square$

The standard form of the Spectral Theorem (Theorem 3.15) states the converse of Lemma 3.9. A proof of it relies on Theorem 3.11. In fact, Theorems 3.11 and 3.15 can be thought of as equivalent versions of the Spectral Theorem.

3.4 Spectral Theorem: General Case

As always, all Hilbert spaces are nonzero and complex. Consider the notation in the paragraph preceding Proposition 3.B. Take the *multiplication operator M_ϕ* in $\mathcal{B}[L^2(\Omega,\mu)]$, which is normal, where ϕ lies in $L^\infty(\Omega,\mu)$, Ω is a nonempty set, and μ is a positive measure on $\mathcal{A}_\Omega$ (cf. Proposition 3.B).

Theorem 3.11. (SPECTRAL THEOREM – FIRST VERSION). *If T is a normal operator on a Hilbert space, then there is a positive measure μ on a σ-algebra $\mathcal{A}_\Omega$ of subsets of a set $\Omega \neq \varnothing$ and a function φ in $L^\infty(\Omega,\mu)$ such that T is unitarily equivalent to the multiplication operator M_φ on $L^2(\Omega,\mu)$, and $\|\varphi\|_\infty = \|T\|$.*

Proof. Let T be a normal operator on a Hilbert space $\mathcal{H}$. We split the proof into two parts. In part (a) we prove the theorem for the case where there exists

a *star-cyclic vector* x for T, which means that there is a vector $x \in \mathcal{H}$ such that $\bigvee_{m,n}\{T^n T^{*m} x\} = \mathcal{H}$ (each index m and n runs over all nonnegative integers; that is, $(m,n) \in \mathbb{N}_0 \times \mathbb{N}_0$). In part (b) we consider the complementary case.

(a) Fix a unit vector $x \in \mathcal{H}$. Take a compact $\varnothing \neq \Omega \subset \mathbb{C}$. Let $P(\Omega)$ be the set of all polynomials $p(\cdot,\cdot)\colon \Omega \times \Omega \to \mathbb{C}$ in λ and $\overline{\lambda}$ (equivalently, $p(\cdot,\overline{\cdot})\colon \Omega \to \mathbb{C}$ such that $\lambda \mapsto p(\lambda,\overline{\lambda})$). Let $C(\Omega)$ be the set of all complex-valued continuous functions on Ω. Thus $P(\Omega) \subset C(\Omega)$. When equipped with the sup-norm, $C(\Omega)$ is a Banach space. The STONE–WEIERSTRASS THEOREM for complex functions (see, e.g., [31, p. 139] or [27, Theorem V.8.1]) says that, if Ω is compact, then

$$P(\Omega)^- = C(\Omega) \quad \text{in the sup-norm topology.}$$

That is, $P(\Omega)$ is dense in the Banach space $(C(\Omega), \|\cdot\|_\infty)$. Consider the functional $\Phi\colon P(\Omega) \to \mathbb{C}$ given, for each $p \in P(\Omega)$, by

$$\Phi(p) = \langle p(T,T^*)x \,;\, x\rangle,$$

which is clearly linear on the linear space $P(\Omega)$. Moreover,

$$|\Phi(p)| \leq \|p(T,T^*)\|\,\|x\|^2 = \|p(T,T^*)\| = r(p(T,T^*))$$

by the Schwarz inequality, since $\|x\| = 1$ and $p(T,T^*)$ is normal (thus normaloid). From now on set $\Omega = \sigma(T)$. According to Theorem 2.8,

$$|\Phi(p)| \,\leq\, r(p(T,T^*)) \,=\, \sup_{\lambda \in \sigma(p(T,T^*))} |\lambda| \,=\, \sup_{\lambda \in \sigma(T)} |p(\lambda,\overline{\lambda})| = \|p\|_\infty,$$

when $P(\sigma(T))$ is viewed as a linear manifold of the Banach space $C(\sigma(T))$ equipped with the sup-norm $\|\cdot\|_\infty$. Thus $\Phi\colon P(\sigma(T)) \to \mathbb{C}$ is bounded too (a contraction, actually, once $|\Phi| \leq 1$). Since $P(\sigma(T))$ is dense in $C(\sigma(T))$ and since $\Phi\colon P(\sigma(T)) \to \mathbb{C}$ is continuous, Φ has a unique bounded linear extension over $C(\sigma(T))$, which we shall denote again by $\Phi\colon C(\sigma(T)) \to \mathbb{C}$.

Claim. $\Phi\colon C(\sigma(T)) \to \mathbb{C}$ is positive in the sense that

$$\Phi(\psi) \geq 0 \text{ for every } \psi \in C(\sigma(T)) \text{ such that } \psi(\lambda) \geq 0 \text{ for every } \lambda \in \sigma(T).$$

Proof. Take any $\psi \in C(\sigma(T))$ such that $\psi(\lambda) \geq 0$ for every $\lambda \in \sigma(T)$. Take an arbitrary $\varepsilon > 0$. Since Φ is continuous and $P(\sigma(T))^- = C(\sigma(T))$, there is a polynomial $p_\varepsilon \in P(\sigma(T))$ such that $p_\varepsilon(\lambda,\overline{\lambda}) \geq 0$ for all $\lambda \in \sigma(T)$, and

$$|\Phi(p_\varepsilon) - \Phi(\psi)| < \varepsilon.$$

The Stone–Weierstrass Theorem for real functions (see, e.g., [31, p. 137] or [27, Theorem V.8.1]) ensures that there exists a polynomial q_ε in the real variables α, β and with real coefficients, $(\alpha,\beta) \mapsto q_\varepsilon(\alpha,\beta) \in \mathbb{R}$, such that

$$|q_\varepsilon(\alpha,\beta)^2 - p_\varepsilon(\lambda,\overline{\lambda})| < \varepsilon \quad \text{for every} \quad \lambda = \alpha + i\beta \in \sigma(T)$$

(recall that $p_\varepsilon(\lambda, \overline{\lambda}) \geq 0$). Now consider the Cartesian decomposition of T, namely, $T = A + iB$, where A and B are commuting self-adjoint operators (see Proposition 1.O). Then $q_\varepsilon(A, B)$ is self-adjoint, and so $q_\varepsilon(A, B)^2$ is a nonnegative operator that commutes with T and T^*. Observe that

$$
\begin{aligned}
|\langle q_\varepsilon(A, B)^2 x \,;\, x\rangle - \Phi(p_\varepsilon)| &= |\langle q_\varepsilon(A, B)^2 x \,;\, x\rangle - \langle p_\varepsilon(T, T^*) x \,;\, x\rangle| \\
&= |\langle (q_\varepsilon(A, B)^2 - p_\varepsilon(T, T^*)) x \,;\, x\rangle| \\
&\leq \|q_\varepsilon(A, B)^2 - p_\varepsilon(T, T^*)\| \\
&= r(q_\varepsilon(A, B)^2 - p_\varepsilon(T, T^*))
\end{aligned}
$$

since $q_\varepsilon(A, B)^2 - p_\varepsilon(T, T^*)$ is normal, thus normaloid (reason: $q_\varepsilon(A, B)^2$ is a self-adjoint operator that commutes with the normal operator $p_\varepsilon(T, T^*)$). So

$$
|\langle q_\varepsilon(A, B)^2 x \,;\, x\rangle - \Phi(p_\varepsilon)| \leq \sup_{\lambda = \alpha + i\beta \in \sigma(T)} |q_\varepsilon(\alpha, \beta)^2 - p_\varepsilon(\lambda, \overline{\lambda})| \leq \varepsilon,
$$

according to Theorem 2.8 again. Hence,

$$
|\langle q_\varepsilon(A, B)^2 x \,;\, x\rangle - \Phi(\psi)| \leq |\langle q_\varepsilon(A, B)^2 x \,;\, x\rangle - \Phi(p_\varepsilon)| + |\Phi(p_\varepsilon) - \Phi(\psi)| \leq 2\varepsilon.
$$

But $0 \leq \langle p_\varepsilon(A, B)^2 x \,;\, x\rangle$, and so $0 \leq \langle p_\varepsilon(A, B)^2 x \,;\, x\rangle \leq 2\varepsilon + \Phi(\psi)$ for every $\varepsilon > 0$, which implies that $0 \leq \Phi(\psi)$, thus completing the proof.

Now take the bounded, linear, positive functional Φ on $C(\Omega)$ (positiveness is understood in the sense of the above Claim) with $\Omega = \sigma(T)$, which is compact. The Riesz Representation Theorem in $C(\Omega)$ (see, e.g., [79, Theorem 2.14]) ensures the existence of a *finite* positive measure μ on the σ-algebra $\mathcal{A}_\Omega = \mathcal{A}_{\sigma(T)}$ of Borel subsets of Ω (where continuous functions are measurable) such that

$$
\Phi(\psi) = \int \psi \, d\mu \quad \text{for every} \quad \psi \in C(\Omega).
$$

Let $U : P(\Omega) \to \mathcal{H}$ be a mapping defined, for each $p \in P(\Omega)$, by

$$
Up = p(T, T^*)x.
$$

Regard $P(\Omega)$ as a linear manifold of the Banach space $L^2(\Omega, \mu)$ equipped with the L^2-norm $\|\cdot\|_2$. It is plain that U is a linear transformation. Also,

$$
\begin{aligned}
\|Up\|_2^2 &= \int |p|^2 \, d\mu = \int p\overline{p} \, d\mu = \Phi(p\overline{p}) = \langle p(T, T^*)\overline{p(T, T^*)} x \,;\, x\rangle \\
&= \langle p(T, T^*)p(T, T^*)^* x \,;\, x\rangle = \|p(T, T^*)^* x\|^2 = \|p(T, T^*)x\|^2
\end{aligned}
$$

for every $p \in P(\Omega)$ (since $p(T, T^*)$ is normal), which says that U is an isometry, thus injective, and so it has an inverse on its range

$$
\mathcal{R}(U) = \{p(T, T^*)x \colon p \in P(\Omega)\} \subseteq \mathcal{H},
$$

say $U^{-1} : \mathcal{R}(U) \to P(\Omega)$. For each polynomial p in $P(\Omega)$ let q be the polynomial in $P(\Omega)$ defined by $q(\lambda, \overline{\lambda}) = \lambda p(\lambda, \overline{\lambda})$ for every $\lambda \in \Omega$. Let $\varphi : \Omega \to \Omega \subset \mathbb{C}$

be the identity function, $\varphi(\lambda) = \lambda$ for every $\lambda \in \Omega$, which is bounded (i.e., $\varphi \in L^\infty(\Omega, \mu)$ because Ω is bounded). In this case we may write

$$q = \varphi p \quad \text{and} \quad q(T, T^*)x = Tp(T, T^*)x.$$

Therefore,

$$U^{-1}TUp = U^{-1}Tp(T, T^*)x = U^{-1}q(T, T^*)x$$
$$= U^{-1}Uq = q = \varphi p = M_\varphi p,$$

for every $p \in P(\Omega)$. Since $P(\Omega)$ is dense in the Banach space $(C(\Omega), \|\cdot\|_\infty)$, and since Ω is a compact set in $\mathbb{C}$ and μ is a finite positive measure on $\mathcal{A}_\Omega$, it follows that $P(\Omega)$ is also dense in the normed space $(C(\Omega), \|\cdot\|_2)$, which in turn is a dense linear manifold of the Banach space $(L^2(\Omega, \mu), \|\cdot\|_2)$, and hence $P(\Omega)$ is dense in $(L^2(\Omega, \mu), \|\cdot\|_2)$. That is,

$$P(\Omega)^- = L^2(\Omega, \mu) \quad \text{in the } L^2\text{-norm topology}.$$

Moreover, since U is a linear isometry of $P(\Omega)$ onto $\mathcal{R}(U)$, which are linear manifolds of the Hilbert spaces $L^2(\Omega, \mu)$ and $\mathcal{H}$, it follows that U extends by continuity to a unique unitary transformation, also denoted by U, of the Hilbert space $P(\Omega)^- = L^2(\Omega, \mu)$ onto the range of U, $\mathcal{R}(U) \subseteq \mathcal{H}$. (In fact, it extends onto the closure of $\mathcal{R}(U)$, but $\mathcal{R}(U)$ is closed in $\mathcal{H}$ because every isometry of a Banach space into a normed space has a closed range.) Thus if x is a star-cyclic vector for T; that is, if $\bigvee_{m,n}\{T^nT^{*m}x\} = \mathcal{H}$, then

$$\mathcal{R}(U) = \big\{p(T, T^*)x\colon p \in P(\Omega)\big\}^- = \bigvee_{m,n}\{T^nT^{*m}x\} = \mathcal{H},$$

and so U is a unitary transformation in $\mathcal{B}[L^2(\Omega, \mu), \mathcal{H}]$ such that

$$U^{-1}TU = M_\varphi.$$

Then T on $\mathcal{H}$ is unitarily equivalent to the multiplication operator M_φ on $L^2(\Omega, \mu)$ for the identity function $\varphi\colon \Omega \to \Omega \subset \mathbb{C}$ on $\Omega = \sigma(T)$ (i.e., $\varphi(\lambda) = \lambda$ for every $\lambda \in \Omega$). Therefore, since T is normal (thus normaloid) and unitarily equivalent to M_φ (thus with the same norm of M_φ), it follows that $\|\varphi\|_\infty = \|T\|$. Indeed, according to Proposition 3.B,

$$\|\varphi\|_\infty = \operatorname{ess\,sup}|\varphi| \le \sup_{\lambda \in \sigma(T)}|\lambda| = r(T) = \|T\| = \|M_\varphi\| = \|\varphi\|_\infty.$$

(b) On the other hand, suppose there is no star-cyclic vector for T. Assume that T and $\mathcal{H}$ are nonzero to avoid trivialities. Since T is normal, the nontrivial subspace $\mathcal{M}_x = \bigvee_{m,n}\{T^nT^{*m}x\}$ of $\mathcal{H}$ reduces T for each unit vector $x \in \mathcal{H}$. Consider the set $\Im = \{\bigoplus \mathcal{M}_x\colon \text{unit } x \in \mathcal{H}\}$ of all *orthogonal* direct sums of these subspaces, which is partially ordered (in the inclusion ordering). $\Im$ is not empty (if a unit $y \in \mathcal{M}_x{}^\perp$, then $\mathcal{M}_y = \bigvee_{m,n}\{T^nT^{*m}y\} \subseteq \mathcal{M}_x{}^\perp$ because $\mathcal{M}_x{}^\perp$ reduces T, and so $\mathcal{M}_x \perp \mathcal{M}_y$). Moreover, every chain in $\Im$ has an upper bound

in $\Im$ (the union of all orthogonal direct sums in a given chain of orthogonal direct sums in $\Im$ is again an orthogonal direct sum in $\Im$). Thus Zorn's Lemma ensures that $\Im$ has a maximal element, say $\mathcal{M} = \bigoplus \mathcal{M}_x$, which coincides with $\mathcal{H}$ (otherwise it would not be maximal since $\mathcal{M} \oplus \mathcal{M}^\perp = \mathcal{H}$). Summing up:

> There exists an indexed family $\{x_\gamma\}_{\gamma \in \Gamma}$ of unit vectors in $\mathcal{H}$ generating a (similarly indexed) collection $\{\mathcal{M}_\gamma\}_{\gamma \in \Gamma}$ of orthogonal nontrivial subspaces of $\mathcal{H}$ such that each $\mathcal{M}_\gamma = \bigvee_{m,n} \{T^n T^{*m} x_\gamma\}$ reduces T, each x_γ is star-cyclic for $T|_{\mathcal{M}_\gamma}$, and $\mathcal{H} = \bigoplus_{\gamma \in \Gamma} \mathcal{M}_\gamma$.

Each restriction $T|_{\mathcal{M}_\gamma}$ is a normal operator in $\mathcal{B}[\mathcal{M}_\gamma]$ (cf. Proposition 1.Q). Thus, by part (a), for each $\gamma \in \Gamma$ there is a positive finite measure μ_γ on the σ-algebra $\mathcal{A}_{\Omega_\gamma}$ of Borel subsets of $\Omega_\gamma = \sigma(T|_{\mathcal{M}_\gamma})$ such that $T|_{\mathcal{M}_\gamma}$ in $\mathcal{B}[\mathcal{M}_\gamma]$ is unitarily equivalent to the multiplication operator M_{φ_γ} in $\mathcal{B}[L^2(\Omega_\gamma, \mu_\gamma)]$, where each $\varphi_\gamma \colon \Omega_\gamma \to \Omega_\gamma \subset \mathbb{C}$ is the identity function on Ω_γ ($\varphi_\gamma(\lambda) = \lambda$ for every $\lambda \in \Omega_\gamma$), which is bounded ($\varphi_\gamma \in L^\infty(\Omega_\gamma, \mu_\gamma)$ since each Ω_γ is compact in $\mathbb{C}$. Thus there exists a unitary transformation U_γ in $\mathcal{B}[\mathcal{M}_\gamma, L^2(\Omega_\gamma, \mu_\gamma)]$ such that $U_\gamma T|_{\mathcal{M}_\gamma} = M_{\varphi_\gamma} U_\gamma$ for each index $\gamma \in \Gamma$. Hence the unitary $U = \bigoplus_{\gamma \in \Gamma} U_\gamma$ in $\mathcal{B}[\bigoplus_{\gamma \in \Gamma} \mathcal{M}_\gamma, \bigoplus_{\gamma \in \Gamma} L^2(\Omega_\gamma, \mu_\gamma)]$ is such that $(\bigoplus_{\gamma \in \Gamma} U_\gamma)(\bigoplus_{\gamma \in \Gamma} T|_{\mathcal{M}_\gamma}) = \bigoplus_{\gamma \in \Gamma} U_\gamma T|_{\mathcal{M}_\gamma} = \bigoplus_\gamma M_{\varphi_\gamma} U_\gamma = (\bigoplus_{\gamma \in \Gamma} M_{\varphi_\gamma})(\bigoplus_{\gamma \in \Gamma} U_\gamma)$. Therefore,

$$T = \bigoplus_{\gamma \in \Gamma} T|_{\mathcal{M}_\gamma} \text{ in } \mathcal{B}[\mathcal{H}] = \mathcal{B}[\bigoplus_{\gamma \in \Gamma} \mathcal{M}_\gamma] \text{ is}$$
$$\text{unitarily equivalent to } \bigoplus_{\gamma \in \Gamma} M_{\varphi_\gamma} \text{ in } \mathcal{B}[\bigoplus_{\gamma \in \Gamma} L^2(\Omega_\gamma, \mu_\gamma)].$$

Consider the disjoint union of $\{\Omega_\gamma\}$, which is obtained by reindexing the elements of each Ω_γ as follows. For each $\gamma \in \Gamma$ write $\Omega_\gamma = \{\lambda_\delta(\gamma)\}_{\delta \in \Delta_\gamma}$ so that, if there is a complex number λ in $\Omega_\alpha \cap \Omega_\beta$ for $\alpha \neq \beta$ in Γ, then the same complex number λ is represented as $\lambda_{\delta'}(\alpha) \in \Omega_\alpha$ for some $\delta' \in \Delta_\alpha$ and $\lambda_{\delta''}(\beta) \in \Omega_\beta$ for some $\delta'' \in \Delta_\beta$, and these representations are considered as distinct elements. Therefore the sets Ω_α and Ω_β become disjoint. (Regard the sets in $\{\Omega_\gamma\}$ as subsets of distinct copies of $\mathbb{C}$ so that, for distinct indices $\alpha, \beta \in \Gamma$, the sets Ω_α and Ω_β are disjoint.) Set $\Omega = \bigcup_{\gamma \in \Gamma} \Omega_\gamma$. The disjoint union Ω — unlike the ordinary union — can viewed as the union of all $\sigma(T|_{\mathcal{M}_\gamma})$ considering the multiplicity of each point in $\bigcup_{\gamma \in \Gamma} \sigma(T|_{\mathcal{M}_\gamma})$, disregarding possible overlappings. Note that $\sigma(T|_{\mathcal{M}_\gamma}) \subseteq \sigma(T)$ since each $\mathcal{M}_\gamma$ reduces T (cf. Proposition 2.F(b)), and so $\Omega \subseteq \bigcup_{\gamma \in \Gamma} \sigma(T)$ is bounded. Let $\mathcal{A}_\Omega$ be the σ-algebra consisting of all subsets of Ω of the form $\Lambda = \bigcup_{\gamma \in \Gamma} \Lambda_\gamma$, where each Λ_γ is a Borel subset of Ω_γ (i.e., $\Lambda_\gamma \in \mathcal{A}_{\Omega_\gamma}$). These are referred to as the Borel subsets of Ω. Let $\mu_\gamma \colon \mathcal{A}_{\Omega_\gamma} \to \mathbb{R}$ be the finite measure on each $\mathcal{A}_{\Omega_\gamma}$ whose existence was ensured in part (a). Consider the Borel subsets $\Lambda = \bigcup_{j \in \mathbb{J}} \Lambda_j$ of Ω made up of disjoint countable unions of Borel subsets of Ω_j (i.e., $\Lambda_j \in \mathcal{A}_{\Omega_j}$), and let μ be the measure on $\mathcal{A}_\Omega$ such that $\mu(\Lambda) = \sum_{j \in \mathbb{J}} \mu_j(\Lambda_j)$ for every $\Lambda = \bigcup_{j \in \mathbb{J}} \Lambda_j$ in $\mathcal{A}_\Omega$ with $\{\Lambda_j\}$ being any disjoint collection of sets with each $\Lambda_j \in \mathcal{A}_{\Omega_j}$, for any countable subset $\mathbb{J}$ of Γ. (For a proper subset $\mathbb{J}$ of Γ we identify $\bigcup_{j \in \mathbb{J}} \Lambda_j$ with $\bigcup_{\gamma \in \Gamma} \Lambda_\gamma$ when $\Lambda_\gamma = \varnothing$ for every $\gamma \notin \mathbb{J}$.) Note that μ is positive since

each μ_j is. (Indeed, $\Lambda_j = \Omega_j$ so that $\mu_j(\Lambda_j) > 0$ for at least one j in one $\mathbb{J}$.)

Consider the measure space $(\Omega, \mathcal{A}_\Omega, \mu)$.

Let $\varphi \colon \Omega \to \mathbb{C}$ be the function obtained from the functions $\varphi_\gamma \colon \Omega_\gamma \to \Omega_\gamma \subset \mathbb{C}$ as follows. If $\lambda \in \bigcup_{\gamma \in \Gamma} \Omega_\gamma$, then $\lambda \in \Omega_\beta$ for some $\beta \in \Gamma$, and so set $\varphi(\lambda) = \varphi_\beta(\lambda) = \lambda$ (each φ_γ is the identity function on Ω_γ). Since $\Omega = \bigcup_{\gamma \in \Gamma} \Omega_\gamma$, where $\Omega_\gamma = \sigma(T|_{\mathcal{M}_\gamma}) \subseteq \sigma(T)$, it follows that $\varphi \in L^\infty(\Omega, \mu)$ with

$$\|\varphi\|_\infty = \operatorname{ess\,sup} |\varphi| \leq \sup_{\lambda \in \Omega} |\varphi(\lambda)| \leq \sup_{\lambda \in \sigma(T)} |\lambda| = r(T) \leq \|T\|.$$

(Actually, $r(T) = \|T\|$ because T is normaloid.) Now we show that

$$\bigoplus_{\gamma \in \Gamma} M_{\varphi_\gamma} \text{ in } \mathcal{B}[\bigoplus_\gamma L^2(\Omega_\gamma, \mu_\gamma)] \text{ is}$$
$$\text{unitarily equivalent to } M_\varphi \text{ in } \mathcal{B}[L^2(\Omega, \mu)].$$

First note that the Hilbert spaces $L^2(\Omega, \mu)$ and $\bigoplus_{\gamma \in \Gamma} L^2(\Omega_\gamma, \mu_\gamma)$ are unitarily equivalent. Actually, recall that direct sums and topological sums of families of orthogonal subspaces of a Hilbert space are unitarily equivalent. Thus $\bigoplus_{\gamma \in \Gamma} L^2(\Omega_\gamma, \mu_\gamma) \cong (\sum_{\gamma \in \Gamma} L^2(\Omega_\gamma, \mu_\gamma))^-$, which means that there is a unitary transformation $V \colon \bigoplus_{\gamma \in \Gamma} L^2(\Omega_\gamma, \mu_\gamma) \to (\sum_{\gamma \in \Gamma} L^2(\Omega_\gamma, \mu_\gamma))^-$. Next consider the mapping $W \colon \sum_{\gamma \in \Gamma} L^2(\Omega_\gamma, \mu_\gamma) \to L^2(\bigcup_{\gamma \in \Gamma} \Omega_\gamma, \mu)$ that assigns to each $\psi = \sum_{\gamma \in \Gamma} \psi_\gamma$ in $\sum_{\gamma \in \Gamma} L^2(\Omega_\gamma, \mu_\gamma)$, with each ψ_γ in $L^2(\Omega_\gamma, \mu_\gamma)$, the function $W\psi$ in $L^2(\bigcup_{\gamma \in \Gamma} \Omega_\gamma, \mu)$ defined as follows: if $\lambda \in \bigcup_{\gamma \in \Gamma} \Omega_\gamma$, then $\lambda \in \Omega_\beta$ for some $\beta \in \Gamma$ and $(W\psi)(\lambda) = \psi_\beta(\lambda)$. The mapping W is linear and surjective. Since $\|W\psi\|^2 = \int |W\psi|^2 \, d\mu = \sum_{\beta \in \Gamma} \int |\psi_\beta|^2 \, d\mu_\beta = \sum_{\beta \in \Gamma} \|\psi_\beta\|^2 = \|\psi\|^2$, it follows that W is an isometry as well, and so it is a unitary transformation of $(\sum_{\gamma \in \Gamma} L^2(\Omega_\gamma, \mu_\gamma))$ onto $L^2(\bigcup_{\gamma \in \Gamma} \Omega_\gamma, \mu)$. Then it extends by continuity to a unitary transformation, also denoted by W, over the whole Hilbert space $(\sum_{\gamma \in \Gamma} L^2(\Omega_\gamma, \mu_\gamma))^-$ onto the Hilbert space $L^2(\bigcup_{\gamma \in \Gamma} \Omega_\gamma, \mu)$. In other words, $W \colon (\sum_{\gamma \in \Gamma} L^2(\Omega_\gamma, \mu_\gamma))^- \to L^2(\bigcup_{\gamma \in \Gamma} \Omega_\gamma, \mu) = L^2(\Omega, \mu)$ is unitary. Therefore, $(\sum_{\gamma \in \Gamma} L^2(\Omega_\gamma, \mu_\gamma))^- \cong L^2(\Omega, \mu)$, and hence, by transitivity,

$$\bigoplus_{\gamma \in \Gamma} L^2(\Omega_\gamma, \mu_\gamma) \cong L^2(\Omega, \mu).$$

Now take an arbitrary $\psi = \sum_{\gamma \in \Gamma} \psi_\gamma$ in $\sum_{\gamma \in \Gamma} L^2(\Omega_\gamma, \mu_\gamma)$, and an arbitrary $\lambda \in \Omega = \bigcup_{\gamma \in \Gamma} \Omega_\gamma$ so that $\lambda \in \Omega_\beta$ for some $\beta \in \Gamma$. Recall that $\varphi|_{\Omega_\beta} = \varphi_\beta$ (which is the identity function on Ω_β). Thus $W\psi(\lambda) = \psi_\beta(\lambda)$, and therefore $(M_\varphi W\psi)(\lambda) = (M_\varphi \psi_\beta)(\lambda) = \varphi(\lambda)\psi_\beta(\lambda) = \varphi|_{\Omega_\beta}(\lambda)\psi_\beta(\lambda) = (\varphi_\beta \psi_\beta)(\lambda)$. On the other hand, since $V^{-1}\psi = \bigoplus_{\gamma \in \Gamma} \psi_\gamma$, it follows that $(\bigoplus_{\gamma \in \Gamma} M_{\varphi_\gamma})V^{-1}\psi = \bigoplus_{\gamma \in \Gamma} M_{\varphi_\gamma}\psi_\gamma = \bigoplus_{\gamma \in \Gamma} \varphi_\gamma \psi_\gamma$, and so $V(\bigoplus_{\gamma \in \Gamma} M_{\varphi_\gamma})V^{-1}\psi = \sum_{\gamma \in \Gamma} \varphi_\gamma \psi_\gamma$. Hence, $(WV(\bigoplus_{\gamma \in \Gamma} M_{\varphi_\gamma})V^{-1}\psi)(\lambda) = (\sum_{\gamma \in \Gamma} \varphi_\gamma \psi_\gamma)_\beta(\lambda) = (\varphi_\beta \psi_\beta)(\lambda)$. Then $M_\varphi W = WV(\bigoplus_{\gamma \in \Gamma} M_{\varphi_\gamma})V^{-1}$ on $\sum_{\gamma \in \Gamma} L^2(\Omega_\gamma, \mu_\gamma)$, which extends by continuity over all $(\sum_{\gamma \in \Gamma} L^2(\Omega_\gamma, \mu_\gamma))^-$. Thus, for the unitary transformation WV,

$$WV\Big(\bigoplus_{\gamma\in\Gamma}M_{\varphi_\gamma}\Big) = M_\varphi WV,$$

and so $\bigoplus_{\gamma\in\Gamma}M_{\varphi_\gamma}$ is unitarily equivalent to M_φ (i.e., $\bigoplus_\gamma M_{\varphi_\gamma}\cong M_\varphi$) through the unitary WV. Then, $T\cong M_\varphi$ by transitivity, since $T\cong\bigoplus_{\gamma\in\Gamma}M_{\varphi_\gamma}$. That is,

$$T \text{ in } \mathcal{B}[\mathcal{H}] \text{ is unitarily equivalent to } M_\varphi \text{ in } \mathcal{B}[L^2(\Omega,\mu)].$$

Finally, since $\|\varphi\|_\infty \le \|T\|$ and $T\cong\mathcal{M}_\varphi$, it follows that (cf. Proposition 3.B),

$$\|\varphi\|_\infty \le \|T\| = \|M_\varphi\| \le \|\varphi\|_\infty. \qquad\square$$

Remark. Note that $\varphi\colon\Omega\to\mathbb{C}$ *can be viewed as a sort of identity function* $(\varphi(\lambda)=\lambda\in\mathbb{C}$ *for every* $\lambda\in\Omega)$ *including multiplicity* (and so it may not be injective), *which takes disjoint unions in* $\mathcal{A}_\Omega$ *into ordinary unions in* $\mathcal{A}_{\sigma(T)}$, *so that* $\varphi(\Omega)^-=\sigma(T)$ (by Proposition 2.F(d) since $T|_{\mathcal{M}_\gamma}$ is normal), *and each* $\tilde\Lambda\in\mathcal{A}_{\sigma(T)}$ *is such that* $\tilde\Lambda^-=\varphi(\Lambda)^-$ *for some* $\Lambda\in\mathcal{A}_\Omega$, *and hence* Ω *is bounded.*

The following commutative diagram illustrates the proof of part (b).

$$
\begin{array}{ccc}
\mathcal{H} & \xrightarrow{\ T\ } & \mathcal{H} \\
\downarrow{\scriptstyle U} & & \downarrow{\scriptstyle U} \\
\bigoplus_{\gamma\in\Gamma}L^2(\Omega_\gamma,\mu_\gamma) & & \bigoplus_{\gamma\in\Gamma}L^2(\Omega_\gamma,\mu_\gamma) \\
\downarrow{\scriptstyle V} & & \downarrow{\scriptstyle V} \\
\big(\sum_{\gamma\in\Gamma}L^2(\Omega_\gamma,\mu_\gamma)\big)^- & & \big(\sum_{\gamma\in\Gamma}L^2(\Omega_\gamma,\mu_\gamma)\big)^- \\
\downarrow{\scriptstyle W} & & \downarrow{\scriptstyle W} \\
L^2(\Omega,\mu) & \xrightarrow{\ M_\varphi\ } & L^2(\Omega,\mu).
\end{array}
$$

Definition 3.12. A vector $x\in\mathcal{H}$ is a *cyclic vector* for an operator T in $\mathcal{B}[\mathcal{H}]$ if $\mathcal{H}$ is the smallest invariant subspace for T that contains x. If T has a cyclic vector, then it is called a *cyclic operator*. A vector $x\in\mathcal{H}$ is a *star-cyclic vector* for an operator T in $\mathcal{B}[\mathcal{H}]$ if $\mathcal{H}$ is the smallest reducing subspace for T that contains x. If T has a star-cyclic vector, then it is called a *star-cyclic operator*.

Observe that $x\in\mathcal{H}$ *is a cyclic vector for an operator* $T\in\mathcal{B}[\mathcal{H}]$ *if and only if* $\bigvee_n\{T^nx\}=\mathcal{H}$. That is, if and only if $\{p(T)x\colon p \text{ is a polynomial}\}^-=\mathcal{H}$; which means that $\{Sx\colon S\in\mathcal{P}(T)\}^-=\mathcal{H}$, where $\mathcal{P}(T)$ is the algebra of all polynomials in T with complex coefficients. Similarly, it can also be verified that $x\in\mathcal{H}$ *is a star-cyclic vector for an operator* $T\in\mathcal{B}[\mathcal{H}]$ *if and only if* $\{Sx\colon S\in\mathcal{C}^*(T)\}^-=\mathcal{H}$, where $\mathcal{C}^*(T)$ stands for the C*-algebra generated by T, which is the smallest C*-algebra of operators from $\mathcal{B}[\mathcal{H}]$ containing T and the identity I. Recall that $\mathcal{C}^*(T)=\mathcal{P}(T,T^*)^-$, the closure in $\mathcal{B}[\mathcal{H}]$ of the set of all polynomials in two (noncommuting) variables T and T^* (in any order) with complex coefficients. (See, e.g., [27, Proposition IX.3.2] and [5, p. 1].)

Remark. Reducing subspaces are invariant. Thus *a cyclic vector is star-cyclic*, and so a cyclic operator is star-cyclic. A normed space is separable if and only if it is spanned by a countable set. So, the subspace of $\mathcal{H}$, $\{Sx \colon S \in \mathcal{P}(T, T^*)\}^- = \bigvee \left(\bigcup_{m,n} \{T^m T^{*n} x\} \cup \{T^{*n} T^m x\} \right)$, is separable, and so is $\{Sx \colon S \in \mathcal{C}^*(T)\}^- = \{Sx \colon S \in \mathcal{P}(T, T^*)^-\}^-$. Hence, *if $T \in \mathcal{B}[\mathcal{H}]$ is a star-cyclic operator, then $\mathcal{H}$ is separable.* (See, e.g., [66, Proposition 4.9] and [27, Proposition IX.3.3].) Then,

$$T \in \mathcal{B}[\mathcal{H}] \text{ is cyclic} \implies T \in \mathcal{B}[\mathcal{H}] \text{ is star-cyclic} \implies \mathcal{H} \text{ is separable.}$$

If T is *normal*, then $\bigvee_{m,n} \{T^n T^{*m} x\}$ is the smallest subspace of $\mathcal{H}$ that contains x and reduces T, so that $x \in \mathcal{H}$ is a *star-cyclic vector* for a *normal* operator T if and only if $\bigvee_{m,n} \{T^n T^{*m} x\} = \mathcal{H}$ (cf. proof of Theorem 3.11). An important result from [19] says that *if T is a normal operator, then T is cyclic whenever it is star-cyclic*, and so

$$T \in \mathcal{B}[\mathcal{H}] \text{ is normal and cyclic} \iff T \in \mathcal{B}[\mathcal{H}] \text{ is normal and star-cyclic.}$$

It can also be shown along this line that *x is a cyclic vector for a normal operator T if and only if it is a cyclic vector for T^** [50, Problem 164].

Consider the proof of Theorem 3.11. Part (a) deals with the case where the normal operator T is star-cyclic, while part (b) deals with the case where T is not star-cyclic. In the former case, $\mathcal{H}$ must be separable by the above Remark. We show next that *if $\mathcal{H}$ is separable, then the measure in* Theorem 3.11 *can be taken to be finite*. Indeed, the positive measure $\mu \colon \mathcal{A}_\Omega \to \mathbb{R}$ constructed in part (a) is finite. However, the positive measure $\mu \colon \mathcal{A}_\Omega \to \overline{\mathbb{R}}$ constructed in part (b) may fail to be even σ-finite (see, e.g., [50, p. 66]). But finiteness can be restored by separability.

Corollary 3.13. *If T is a normal operator on a* separable *Hilbert space $\mathcal{H}$, then there is a* finite *positive measure μ on a σ-algebra $\mathcal{A}_\Omega$ of subsets of a non-empty set Ω and a function φ in $L^\infty(\Omega, \mu)$ such that T is unitarily equivalent to the multiplication operator M_φ on $L^2(\Omega, \mu)$.*

Proof. Consider the argument in part (b) of the proof of Theorem 3.11. Since the collection $\{\mathcal{M}_\gamma\}$ is made up of orthogonal subspaces, we may take one unit vector from each subspace by the Axiom of Choice to get an orthonormal set $\{e_\gamma\}$ of unit vectors, which is in a one-to-one correspondence with the set $\{\mathcal{M}_\gamma\}$. Hence $\{\mathcal{M}_\gamma\}$ and $\{e_\gamma\}$ have the same cardinality. Since $\{e_\gamma\}$ is an orthonormal set, its cardinality is not greater than the cardinality of any orthonormal basis for $\mathcal{H}$, and so the cardinality of $\{\mathcal{M}_\gamma\}$ does not surpass the (Hilbert) dimension of $\mathcal{H}$. If $\mathcal{H}$ is separable, then $\{\mathcal{M}_\gamma\}$ is countable. In this case, for notational convenience, replace the index γ in the index set Γ with an integer index n in a countable index set, say $\mathbb{N}$, throughout the proof of part (b). Since the orthogonal collection $\{\mathcal{M}_n\}$ of subspaces is now countable, it follows that the family of sets $\{\Omega_n\}$, each $\Omega_n = \sigma(T|_{\mathcal{M}_n})$, is countable. Recall that each positive measure $\mu_n \colon \mathcal{A}_{\Omega_n} \to \mathbb{R}$ inherited from part (a) is finite. Take

the positive measure μ on $\mathcal{A}_\Omega$ defined in part (b). Thus $\mu(\Lambda) = \sum_{j \in \mathbb{J}} \mu_j(\Lambda_j)$ for every set $\Lambda = \bigcup_{j \in \mathbb{J}} \Lambda_j$ in $\mathcal{A}_\Omega$ (each Λ_j in $\mathcal{A}_{\Omega_j}$ so that $\Lambda_j \subseteq \Omega_j$), where $\mathbb{J}$ is any subset of $\mathbb{N}$. But now $\Omega = \bigcup_{n \in \mathbb{N}} \Omega_n$ and so $\mu(\Omega) = \sum_{n \in \mathbb{N}} \mu_n(\Omega_n) = \sum_{n \in \mathbb{N}} \mu(\Omega_n)$, where $\mu(\Omega_n) = \mu_n(\Omega_n) < \infty$ for each $n \in \mathbb{N}$. (Again, we are identifying each Ω_n with $\bigcup_{m \in \mathbb{N}} \Omega_m$ when $\Omega_m = \varnothing$ for every $m \neq n$.) Conclusion: If $\mathcal{H}$ is separable, then μ is σ-finite. However, we can actually get a finite measure. Indeed, without loss of generality, multiply each original measure μ_n by a positive scalar. Then we may assume that $\sum_{n \in \mathbb{N}} \mu_n(\Omega_n) < \infty$ (e.g., take $0 < \mu_n(\Omega_n) \leq 2^{-n}$ for each $n \in \mathbb{N}$). In this case, $\mu(\Omega) = \sum_{n \in \mathbb{N}} \mu_n(\Omega_n) < \infty$, so that $\mu \colon \mathcal{A}_\Omega \to \mathbb{R}$ is now finite (rather than just σ-finite). Thus all measures involved in the proof of Theorem 3.11 are finite whenever $\mathcal{H}$ is separable. $\square$

Remark. Corollary 3.13 forces the Hilbert space $L^2(\Omega, \mu)$, with that particular finite measure μ, to be separable (since it is unitarily equivalent to a separable Hilbert space $\mathcal{H}$). Recall that there exist nonseparable L^2 spaces with finite measures, and also non-σ-finite measures for which L^2 is separable (see, e.g., [21, pp. 174, 192, 376] and [51, pp. 2, 3]).

Corollary 3.14. *Let T be a normal operator on a Hilbert space $\mathcal{H}$ so that $T \cong M_\varphi$ on $L^2(\Omega, \mu)$, where φ lies in $L^\infty(\Omega, \mu)$, with μ being the positive measure on the σ-algebra $\mathcal{A}_\Omega$ of subsets of the nonempty set Ω as in Theorem 3.11. The following assertions are pairwise equivalent.*

(a) *$\mathcal{H}$ is separable.*

(b) *μ is finite.*

(c) *T is star-cyclic.*

(d) *T is cyclic.*

Moreover, if any of the above equivalent assertions holds true, then the constant function $\psi = 1$ ($\psi(\lambda) = 1$ for all $\lambda \in \Omega$) lies in $L^2(\Omega, \mu)$ and is star-cyclic for M_φ, and so M_φ has a star-cyclic vector in $L^\infty(\Omega, \mu)$.

Proof. By Theorem 3.11, $T \cong M_\varphi$. That is, T is unitarily equivalent to the multiplication operator M_φ on $L^2(\Omega, \mu)$, where φ is a function in $L^\infty(\Omega, \mu)$. Let U be a unitary transformation in $\mathcal{B}[L^2(\Omega, \mu), \mathcal{H}]$ such that $U^*TU = M_\varphi$. Observe that ψ in $L^2(\Omega, \mu)$ is a cyclic (star-cyclic) vector for the normal operator M_φ if and only if $x = U\psi$ in $\mathcal{H}$ is a cyclic (star-cyclic) vector for the normal operator T. Indeed, $\bigvee_n \{T^n x\} = U \bigvee_n \{M_\varphi^n \psi\}$ and $\bigvee_{m,n} \{T^n T^{*m} x\} = U \bigvee_{m,n} \{M_\varphi^n M_\varphi^{*m} \psi\}$. Corollary 3.13 says that (a) implies (b). We show next that (b) implies (c). Since $\varphi \colon \Omega \to \mathbb{C}$ is such that $\varphi(\lambda) = \lambda \in \mathbb{C}$ for every $\lambda \in \Omega$, a trivial induction ensures that $M_\varphi^n \psi = \lambda^n \psi$, $M_\varphi^{*m} \psi = \overline{\lambda}^m \psi$, and so $\bigvee_{m,n} \{M_\varphi^n M_\varphi^{*m} \psi\} = \bigvee_{m,n} \{\lambda^n \overline{\lambda}^m \psi\}$, for every λ in Ω and every $\psi \in L^2(\Omega, \mu)$. If μ is finite, then the constant function 1 (i.e., $1(\lambda) = 1$ for all $\lambda \in \Omega$) lies in $L^2(\Omega, \mu)$. Set $\psi = 1$ in $L^2(\Omega, \mu)$, and let $P(\Omega)$ denote the set of all polynomials $p(\cdot, \cdot) \colon \Omega \times \Omega \to \mathbb{C}$ in λ and $\overline{\lambda}$ (equivalently, $p(\cdot, \overline{\cdot}) \colon \Omega \to \mathbb{C}$ such that $\lambda \mapsto p(\lambda, \overline{\lambda})$). Thus (cf. proof of Theorem 3.11),

$$L^2(\Omega, \mu) \;=\; P(\Omega)^- \;=\; \bigvee_{m,n} \{\lambda^n \overline{\lambda}^m\} \;=\; \bigvee_{m,n} \{M_\varphi^n M_\varphi^{*m} \psi\}$$

so that $\psi = 1$ is a star-cyclic vector for M_φ. Thus (b) implies (c). The Remark preceding Corollary 3.13 ensures that (c) $\Longrightarrow$ (d) $\Longrightarrow$ (a). $\qquad\square$

Remark. Let K be a compact subset of $\mathbb{C}$. The Stone–Weierstrass Theorem (cf. proof of Theorem 3.11) ensures that $P(K)^- = C(K)$ in the sup-norm, where $C(K)$ is the set of all complex-valued continuous functions on K and $P(K)$ is the set of all polynomials $p(\cdot,\cdot) \colon K \times K \to \mathbb{C}$ in λ and $\overline{\lambda}$ (or, $p(\cdot,\bar{\cdot}) \colon K \to \mathbb{C}$ such that $\lambda \mapsto p(\lambda, \overline{\lambda})$). Let $P'(K)$ denote the set of all polynomials $p(\cdot) \colon K \to \mathbb{C}$ in one variable λ. The Lavrentiev Theorem says that *if K is compact, then $P'(K)^- = C(K)$ in the sup-norm if and only if $\mathbb{C} \backslash K$ is connected and $K^\circ = \varnothing$* (i.e., *if and only if the compact set K has empty interior and no holes*), which extends the classical Weierstrass Approximation Theorem for (complex-valued) polynomials on compact intervals of the real line, viz., $P'([\alpha,\beta])^- = C([\alpha,\beta])$ *in the sup-norm* (see [21, p. 18], or [27, Proposition VII.5.3] and [28, Theorem V.14.20]; for real-valued functions the Weierstrass Theorem holds for every compact set in the real line — see [31, p. 139]). This underlines the difference between cyclic and star-cyclic vectors. Note that *if T is normal and $P'(\sigma(T))^- = C(\sigma(T))$, then T is reductive* (Proposition 3.J). Bram's result [19] not only shows that normal operators are cyclic if and only if they are star-cyclic, but also shows that, under the hypotheses of Corollary 3.14, M_φ has a cyclic vector in $L^\infty(\Omega, \mu)$, with $\Omega = \sigma(T)$ as in part (a) of the proof of Theorem 3.11 [28, Theorem V.14.21].

Theorem 3.15. (Spectral Theorem – second version). *If $T \in \mathcal{B}[\mathcal{H}]$ is normal, then there is a unique spectral measure $E \colon \mathcal{A}_{\sigma(T)} \to \mathcal{B}[\mathcal{H}]$ such that*

$$T = \int \lambda \, dE_\lambda.$$

If Λ is a nonempty relatively open subset of $\sigma(T)$, then $E(\Lambda) \neq O$.

Proof. We shall split the proof into four parts.

(a) Let μ be a positive measure on a σ-algebra $\mathcal{A}_\Omega$ of Borel subsets of a non-empty bounded set Ω of complex numbers. Consider the multiplication operator $M_\phi \in \mathcal{B}[L^2(\Omega, \mu)]$ for any $\phi \in L^\infty(\Omega, \mu)$, and let $\chi_\Lambda \colon \Omega \to \{0,1\}$ be the characteristic function of $\Lambda \in \mathcal{A}_\Omega$, which is bounded (i.e., $\chi_\Lambda \in L^\infty(\Omega, \mu)$). Set

$$E'(\Lambda) = M_{\chi_\Lambda}$$

for every $\Lambda \in \mathcal{A}_\Omega$. It is readily verified that $E' \colon \mathcal{A}_\Omega \to \mathcal{B}[L^2(\Omega, \mu)]$ is a spectral measure in $L^2(\Omega, \mu)$ (cf. Definition 3.6). Take an arbitrary $f \in L^2(\Omega, \mu)$. Since a characteristic function of a measurable set is a measurable function,

$$\int \chi_\Lambda d\langle E'_\lambda f \,;\, f\rangle = \int \chi_\Lambda \, d\,\pi'_{f,f} = \int_\Lambda d\,\pi'_{f,f} = \pi'_{f,f}(\Lambda) = \langle E'(\Lambda)f \,;\, f\rangle$$

$$= \langle M_{\chi_\Lambda} f \,;\, f\rangle = \int \chi_\Lambda f \overline{f} \, d\mu = \int \chi_\Lambda |f|^2 d\mu$$

for every $\Lambda \in \mathcal{A}_\Omega$. Let Ψ be the collection of all simple functions $\psi \colon \Omega \to \mathbb{C}$ of the form $\psi = \sum_i \lambda_i \chi_{\Lambda_i}$ with respect to all *finite* measurable partitions $\{\Lambda_i\}$ of Ω and all (similarly indexed) sets $\{\lambda_i\}$ of constants in Ω with each $\lambda_i \in \Lambda_i$ (so that $\psi(\lambda_i) = \lambda_i$). Since each $\psi \in \Psi$ is $\mathcal{A}_\Omega$-measurable (and bounded),

$$\int \psi \, d\langle E'_\lambda f \, ; f \rangle = \sum_i \lambda_i \int \chi_{\Lambda_i} d\langle E'_\lambda f \, ; f \rangle = \sum_i \lambda_i \int \chi_{\Lambda_i} |f|^2 d\mu$$

$$= \int \sum_i \lambda_i \chi_{\Lambda_i} |f|^2 \, d\mu = \int \psi \, |f|^2 d\mu$$

for every $\psi \in \Psi$ (by the previously displayed identity), which also holds for the function $\varphi \colon \Omega \to \mathbb{C}$ such that $\varphi(\lambda) = \lambda$. (Reason: $\inf_{\psi \in \Psi} \|\varphi - \psi\|_\infty = 0$ so that $\varphi \in \Psi^- \subset L^\infty(\Omega, \mu)$, thus apply extension by continuity.) Therefore,

$$\int \varphi \, d\langle E'_\lambda f \, ; f \rangle = \int \varphi \, |f|^2 d\mu = \int \varphi \, f \overline{f} \, d\mu = \langle M_\varphi f \, ; f \rangle$$

for every $f \in L^2(\Omega, \mu)$. By the polarization identity (Proposition 1.A) we get $\langle Sf \, ; g \rangle = \frac{1}{4}[\langle S(f+g) \, ; (f+g) \rangle - \langle S(f-g) \, ; (f-g) \rangle + i \langle S(f+ig) \, ; (f+ig) \rangle - i \langle S(f-ig) \, ; (f-ig) \rangle$ for every $f, g \in L^2(\Omega, \mu)$ and $S \in \mathcal{B}[L^2(\Omega, \mu)]$. Thus, replacing S with $E'(\Lambda)$ on the one hand, and with M_φ on the other hand,

$$\int \varphi \, d\langle E'_\lambda f \, ; g \rangle = \langle M_\varphi f \, ; g \rangle$$

for every $f, g \in L^2(\Omega, \mu)$, which means

$$M_\varphi = \int \varphi(\lambda) \, dE'_\lambda = \int \lambda \, dE'_\lambda.$$

Uniqueness of the spectral measure $E' \colon \mathcal{A}_\Omega \to \mathcal{B}[\mathcal{H}]$ is proved as follows. Let $P(\Omega)$ be the set of polynomials $p(\cdot, \bar{\cdot})$ on Ω as defined in the proof of Theorem 3.11. If $E'' \colon \mathcal{A}_\Omega \to \mathcal{B}[\mathcal{H}]$ is a spectral measure in $L^2(\Omega, \mu)$ such that $M_\varphi = \int \lambda \, dE''_\lambda = \int \lambda \, dE'_\lambda$, then $p(T, T^*) = \int p(\lambda, \bar{\lambda}) \, dE''_\lambda = \int p(\lambda, \bar{\lambda}) \, dE'_\lambda$ for every $p \in P(\Omega)$ by Lemma 3.10. This means that $\int p \, d\langle E''_\lambda x; y \rangle = \int p \, d\langle E'_\lambda x; y \rangle$ for every $x, y \in \mathcal{H}$, for every $p \in P(\Omega)$. Fix an arbitrary $x \in \mathcal{H}$. So, in particular,

$$\int p \, d\langle E''_\lambda x; x \rangle = \int p \, d\langle E'_\lambda x; x \rangle$$

for every $p \in P(\Omega)$. Let K be the closure of the union of the supports of the measures E' and E'', which is compact since Ω is bounded, and let $C(K)$ be the set of all complex-valued continuous functions on K. The Stone–Weierstrass Theorem for complex functions says that $P(K)$ is dense in $(C(K), \|\cdot\|_\infty)$, and so $P(K)$ is dense in $(L^2(K, \nu), \|\cdot\|_2)$ for every finite positive measure ν on $\mathcal{A}_\Omega$ with support in K (see proof of Theorem 3.11). Since the positive measures $\langle E'(\cdot) x \, ; x \rangle = \|E'(\cdot) x\|^2$ and $\langle E''(\cdot) x \, ; x \rangle = \|E''(\cdot) x\|^2$ on $\mathcal{A}_\Omega$ are finite, we get

$$\int \chi_\Lambda d\langle E''_\lambda x; x \rangle = \int \chi_\Lambda d\langle E'_\lambda x; x \rangle$$

for the characteristic function χ_Λ of an arbitrary set $\Lambda \in \mathcal{A}_\Omega$. Thus

$$\langle E''(\varLambda)x\,;x\rangle = \int_{\varLambda} d\langle E''_{\lambda}x;x\rangle = \int_{\varLambda} d\langle E'_{\lambda}x;x\rangle = \langle E'(\varLambda)x\,;x\rangle,$$

and so $\langle (E''(\varLambda) - E'(\varLambda))x\,;x\rangle = 0$, for every $\varLambda \in \mathcal{A}_{\varOmega}$. Since $E''(\varLambda) - E'(\varLambda)$ is self-adjoint for every $\varLambda \in \mathcal{A}_{\varOmega}$, and since the above identity holds for every $x \in \mathcal{H}$, it follows that E' is unique. That is (see, e.g., [66, Corollary 5.80]),

$$E''(\varLambda) - E'(\varLambda) = O \quad \text{for every } \varLambda \in \mathcal{A}_{\varOmega}.$$

This completes the proof for the normal operator $M_{\varphi} \in \mathcal{B}[L^2(\varOmega, \mu)]$: *there is a unique spectral measure* $E': \mathcal{A}_{\varOmega} \to \mathcal{B}[L^2(\varOmega, \mu)]$ *such that* $M_{\varphi} = \int \lambda \, dE'_{\lambda}$.

(b) If $T \in \mathcal{B}[\mathcal{H}]$ is normal, then $T \cong M_{\varphi} \in \mathcal{B}[L^2(\varOmega, \mu)]$ (for a positive measure μ on a σ-algebra $\mathcal{A}_{\varOmega}$ of subsets of a bounded set $\varOmega$ — the disjoint union $\bigcup_{\gamma \in \varGamma} \varOmega_{\gamma}$) by Theorem 3.11. Thus there is a unitary $U \in \mathcal{B}[\mathcal{H}, L^2(\varOmega, \mu)]$ such that $T = U^*M_{\varphi}U$. But $E: \mathcal{A}_{\varOmega} \to \mathcal{B}[\mathcal{H}]$ given by $E(\varLambda) = U^*E'(\varLambda)U$ for each $\varLambda \in \mathcal{A}_{\varOmega}$ is a spectral measure in $\mathcal{H}$ (Definition 3.6). Then we get from (a) that

$$\langle Tx\,;y\rangle = \langle M_{\varphi}Ux\,;Uy\rangle = \int \varphi \, d\langle E'_{\lambda}Ux\,;Uy\rangle = \int \varphi \, d\langle E_{\lambda}x\,;y\rangle$$

for every $x, y \in \mathcal{H}$, with $\varphi: \varOmega \to \mathbb{C}$ such that $\varphi(\lambda) = \lambda$, which means

$$T = \int \lambda \, dE_{\lambda}.$$

Since $E(\varLambda) \cong E'(\varLambda)$, uniqueness of E' implies uniqueness of E. Therefore, *there is a unique spectral measure* $E: \mathcal{A}_{\varOmega} \to \mathcal{B}[\mathcal{H}]$ *such that* $T = \int \lambda \, dE_{\lambda}$.

(c) Recall that $\varOmega$ is the disjoint union $\bigcup_{\gamma \in \varGamma} \varOmega_{\gamma}$ with $\varOmega_{\gamma} = \sigma(T|_{\mathcal{M}_{\gamma}}) \subseteq \sigma(T)$ — a set of complex numbers in $\sigma(T)$ that considers multiplicity of each point λ in $\sigma(T)$. Note that, for each $\lambda \in \varOmega$, $\{\lambda\} \in \mathcal{A}_{\varOmega}$ (cf. proof of Theorem 3.11) and $\dim \mathcal{R}(E(\{\lambda\})) = \dim \mathcal{R}(E'(\{\lambda\})) = \dim \mathcal{R}(M_{\chi_{\{\lambda\}}}) = \dim L^2(\{\lambda\}, \mu) \in \{0, 1\}$. We shall now *incorporate the multiplicity of* λ *in* $\dim \mathcal{R}(E(\{\lambda\}))$ as follows. For each $\varLambda$ in $\mathcal{A}_{\sigma(T)}$ set $\cup \varLambda = \bigcup_{\gamma \in \varGamma} \{\varLambda(\gamma) \in \mathcal{A}_{\varOmega_{\gamma}}: \varLambda(\gamma) = \varLambda \cap \varOmega_{\gamma}\}$ in $\mathcal{A}_{\varOmega}$ so that, for each λ in $\sigma(T)$, $\{\lambda\} \in \mathcal{A}_{\sigma(T)}$ and $\cup \{\lambda\} = \bigcup_{\gamma \in \varGamma} \{\lambda(\gamma) \in \varOmega_{\gamma}: \lambda(\gamma) = \lambda\}$ in $\mathcal{A}_{\varOmega}$. Set $\tilde{E}: \mathcal{A}_{\sigma(T)} \to \mathcal{B}[\mathcal{H}]$ such that $\tilde{E}(\varLambda) = \int_{\cup \varLambda} dE_{\lambda} = E(\cup \varLambda)$ for every $\varLambda$ in $\mathcal{A}_{\sigma(T)}$. This defines a spectral measure $\tilde{E}$ on $\mathcal{A}_{\sigma(T)}$ such that $\int_{\sigma(T)} \lambda \, d\tilde{E}_{\lambda} = \int_{\varOmega} \lambda \, dE_{\lambda}$, which is unique since E is unique. In particular, for each λ in $\sigma(T)$ we get $\tilde{E}(\{\lambda\}) = \int_{\cup \lambda} dE_{\lambda} = E(\cup \lambda)$, and so $\dim \mathcal{R}(\tilde{E}(\{\lambda\})) = \dim \mathcal{R}(E(\cup \lambda))$. Therefore, using the same notation for $\tilde{E}$ and E, it follows that *multiplicity of* λ *is incorporated in* $\dim \mathcal{R}(E(\{\lambda\}))$ and, from (b), *there is a unique spectral measure* $E: \mathcal{A}_{\sigma(T)} \to \mathcal{B}[\mathcal{H}]$ *such that* $T = \int \lambda \, dE_{\lambda}$.

(d) We show that *if* $\varnothing \neq \varLambda \in \mathcal{A}_{\sigma(T)}$ *is relatively open in* $\sigma(T)$, *then* $E(\varLambda) \neq O$. Take any $\varnothing \neq \varLambda \in \mathcal{A}_{\sigma(T)}$ and consider the definition of $\cup \varLambda \in \mathcal{A}_{\varOmega}$ in part (c). If $\varLambda$ is relatively open in $\sigma(T)$, then there is a nonempty $\varLambda(\gamma) \subseteq \varLambda$ in $\mathcal{A}_{\sigma(T|_{\mathcal{M}_{\gamma}})}$ relatively open in $\sigma(T|_{\mathcal{M}_{\gamma}})$. It can be verified that each positive measure μ_{γ} on $\mathcal{A}_{\varOmega_{\gamma}}$ (cf. part (b) in the proof of Theorem 3.11) is such that support $(\mu_{\gamma}) = \sigma(T|_{\mathcal{M}_{\gamma}})$. Thus $\mu_{\gamma}(\varLambda(\gamma)) > 0$, and so $\mu(\cup \varLambda) > 0$ (according to the definition

of μ in the proof of Theorem 3.11). However, since $\chi_{\cup\Lambda} \in L^2(\Omega, \mu)$, and since $\langle E'(\Lambda')f\,;f\rangle = \int \chi_{\Lambda'}|f|^2 d\mu$ for each $\Lambda' \in \Omega$ and every $f \in L^2(\Omega, \mu)$,

$$\|E'(\cup\Lambda)\chi_{\cup\Lambda}\|^2 = \langle E'(\cup\Lambda)\chi_{\cup\Lambda}\,;\chi_{\cup\Lambda}\rangle = \int \chi_{\cup\Lambda}|\chi_{\cup\Lambda}|^2 d\mu = \int_{\cup\Lambda} d\mu = \mu(\cup\Lambda).$$

Thus, $\|E'(\cup\Lambda)\| \neq 0$, and therefore $\|\tilde{E}(\Lambda)\| = \|E(\cup\Lambda)\| = \|E'(\cup\Lambda)\| \neq 0$, so that $\tilde{E}(\Lambda) \neq O$. Again, use the same notation for $\tilde{E}$ and E. $\qquad\square$

The representation $T = \int \lambda\, dE_\lambda$ is also referred to as the *spectral decomposition* of the normal operator T (see Section 3.2).

3.5 Fuglede Theorems and Reducing Subspaces

Let $\mathcal{A}_{\sigma(T)}$ be the σ-algebra of Borel subsets of the spectrum $\sigma(T)$ of T. The following lemma sets up the essential features for proving the next theorem.

Lemma 3.16. *Let* $T = \int \lambda\, dE_\lambda$ *be the spectral decomposition of a normal operator* $T \in \mathcal{B}[\mathcal{H}]$ *as in Theorem 3.15. If* $\Lambda \in \mathcal{A}_{\sigma(T)}$, *then*

(a) $E(\Lambda)T = TE(\Lambda)$. *Equivalently,* $\mathcal{R}(E(\Lambda))$ *reduces* T.

(b) *Moreover, if* $\Lambda \neq \varnothing$, *then* $\sigma(T|_{\mathcal{R}(E(\Lambda))}) \subseteq \Lambda^-$.

Proof. Consider the spectral decomposition of T. Take any $\varnothing \neq \Lambda \in \mathcal{A}_{\sigma(T)}$.

(a) Since $E(\Lambda) = \int \chi_\Lambda(\lambda)\, dE_\lambda$ and $T = \int \lambda\, dE_\lambda$, it follows by Lemma 3.8 that

$$E(\Lambda)T = \int \lambda\,\chi_\Lambda(\lambda)\, dE_\lambda = \int \chi_\Lambda(\lambda)\,\lambda\, dE_\lambda = TE(\Lambda),$$

which is equivalent to saying that $\mathcal{R}(E(\Lambda))$ reduces T (by Proposition 1.I).

(b) Let $\mathcal{A}_\Omega$ be the σ-algebra of Borel subsets of Ω as in Theorem 3.11. Take an arbitrary nonempty $\Lambda \in \mathcal{A}_\Omega$. Recall from the proof of Theorem 3.15 that $\int \lambda\, dE_\lambda = T \cong M_\varphi = \int \varphi(\lambda)\, dE'_\lambda$, so that $T = U^*M_\varphi U$ for a unitary $U\colon \mathcal{H} \to L^2(\Omega, \mu)$, where the spectral measures $E'\colon \mathcal{A}_\Omega \to \mathcal{B}[L^2(\Omega, \mu)]$ and $E\colon \mathcal{A}_\Omega \to \mathcal{B}[\mathcal{H}]$ are given by $E'(\Lambda) = M_{\chi_\Lambda}$ and $E(\Lambda) = U^*E'(\Lambda)U$. Thus

$$T|_{\mathcal{R}(E(\Lambda))} \cong M_\varphi|_{\mathcal{R}(M_{\chi_\Lambda})}.$$

Indeed, since $\mathcal{R}(E(\Lambda)) = U^*\mathcal{R}(E'(\Lambda)U)$, we get

$$T|_{\mathcal{R}(E(\Lambda))} = (U^*M_\varphi U)|_{U^*\mathcal{R}(E'(\Lambda)U)} = U^*M_\varphi|_{\mathcal{R}(E'(\Lambda))}U = U^*M_\varphi|_{\mathcal{R}(M_{\chi_\Lambda})}U.$$

Let $\mu|_\Lambda$ be the restriction of μ to the σ-algebra $\mathcal{A}_\Lambda = \wp(\Lambda) \cap \mathcal{A}_\Omega \subseteq \mathcal{A}_\Omega$ so that $\mathcal{R}(M_{\chi_\Lambda})$ is unitarily equivalent to $L^2(\Lambda, \mu|_\Lambda)$; that is,

$$\mathcal{R}(M_{\chi_\Lambda}) = \big\{f \in L^2(\Omega, \mu)\colon f = \chi_\Lambda g \text{ for some } g \in L^2(\Omega, \mu)\big\} \cong L^2(\Lambda, \mu|_\Lambda).$$

Consider the following notation: set $M_{\varphi\chi_\Lambda} = M_\varphi|_{L^2(\Lambda, \mu|_\Lambda)}$ in $\mathcal{B}[L^2(\Lambda, \mu|_\Lambda)]$ (not in $\mathcal{B}[L^2(\Omega, \mu)]$). Since $\mathcal{R}(E'(\Lambda)) = \mathcal{R}(M_{\chi_\Lambda})$ reduces M_φ,

$$M_\varphi|_{\mathcal{R}(M_{\chi_\Lambda})} \cong M_{\varphi\chi_\Lambda}.$$

Hence $T|_{\mathcal{R}(E(\Lambda))} \cong M_{\varphi\chi_\Lambda}$ by transitivity, and therefore (cf. Proposition 2.B),

$$\sigma(T|_{\mathcal{R}(E(\Lambda))}) = \sigma(M_{\varphi\chi_\Lambda}).$$

Since $\Lambda \neq \varnothing$, Proposition 3.B ensures that, for $\varphi\colon \Omega \to \mathbb{C}$ such that $\varphi(\lambda) = \lambda$,

$$\sigma(M_{\varphi\chi_\Lambda}) \subseteq \bigcap \{\varphi(\Delta)^- \in \mathbb{C}\colon \Delta \in \mathcal{A}_\Lambda \text{ and } \mu|_\Lambda(\Lambda\backslash\Delta) = 0\}$$
$$= \bigcap \{\varphi(\Delta \cap \Lambda)^- \in \mathbb{C}\colon \Delta \in \mathcal{A}_\Omega \text{ and } \mu(\Lambda\backslash\Delta) = 0\} \subseteq \varphi(\Lambda)^-.$$

Finally, take an arbitrary $\tilde{\Lambda} \in \mathcal{A}_{\sigma(T)}$ and let $\tilde{E}\colon \mathcal{A}_{\sigma(T)} \to \mathcal{B}[\mathcal{H}]$ be the spectral measure defined by $\tilde{E}(\tilde{\Lambda}) = E(\cup\tilde{\Lambda})$ (cf. part (c) in the proof of Theorem 3.15). Since $\tilde{\Lambda}^- = \varphi(\Lambda)^-$ for some $\Lambda \in \mathcal{A}_\Omega$, and since $\cup(\varphi(\Lambda)^-) \subseteq \Lambda^-$, we get

$$\sigma(T|_{\mathcal{R}(\tilde{E}(\tilde{\Lambda}))}) \subseteq \sigma(T|_{\mathcal{R}(\tilde{E}(\varphi(\Lambda)^-))}) \subseteq \sigma(T|_{\mathcal{R}(E(\Lambda^-))}) \subseteq \varphi(\Lambda^-)^- = \tilde{\Lambda}^-.$$

Again, use the same notation for E and $\tilde{E}$, and for Λ and $\tilde{\Lambda}$. $\square$

The proof of the next theorem follows the same argument used in the proof of Corollary 3.5, now extended to the general case (see [76, Theorem 1.12]).

Theorem 3.17. (Fuglede Theorem). *Let $T = \int \lambda \, dE_\lambda$ be the spectral decomposition of a normal operator in $\mathcal{B}[\mathcal{H}]$. If $S \in \mathcal{B}[\mathcal{H}]$ commutes with T, then S commutes with $E(\Lambda)$ for every $\Lambda \in \mathcal{A}_{\sigma(T)}$.*

Proof. Take any Λ in $\mathcal{A}_{\sigma(T)}$. Suppose $\varnothing \neq \Lambda \neq \sigma(T)$; otherwise the result is trivially verified with $E(\Lambda) = O$ or $E(\Lambda) = I$. Set $\Lambda' = \sigma(T)\backslash\Lambda$ in $\mathcal{A}_{\sigma(T)}$. Let Δ and Δ' be arbitrary nonempty measurable closed subsets of Λ and Λ'. That is, $\varnothing \neq \Delta \subseteq \Lambda$ and $\varnothing \neq \Delta' \subseteq \Lambda'$ lie in $\mathcal{A}_{\sigma(T)}$ and are closed in $\mathbb{C}$; and so they are closed relative to the compact set $\sigma(T)$. Lemma 3.16(a) ensures that

$$E(\Lambda)T|_{\mathcal{R}(E(\Lambda))} = TE(\Lambda)|_{\mathcal{R}(E(\Lambda))} = T|_{\mathcal{R}(E(\Lambda))}\colon \mathcal{R}(E(\Lambda)) \to \mathcal{R}(E(\Lambda))$$

for every $\Lambda \in \mathcal{A}_{\sigma(T)}$. In particular, this holds for Δ and Δ'. Take any operator S on $\mathcal{H}$ such that $ST = TS$. Thus, on $\mathcal{R}(E(\Delta))$,

$$(*) \quad (E(\Delta')SE(\Delta))T|_{\mathcal{R}(E(\Delta))} = (E(\Delta')SE(\Delta))(TE(\Delta)) = E(\Delta')STE(\Delta)$$
$$= E(\Delta')TSE(\Delta) = (TE(\Delta'))(E(\Delta')SE(\Delta))$$
$$= T|_{\mathcal{R}(E(\Delta'))}(E(\Delta')SE(\Delta)).$$

Since the sets Δ and Δ' are closed and nonempty, Lemma 3.16(b) says that

$$\sigma(T|_{\mathcal{R}(E(\Delta))}) \subseteq \Delta \quad \text{and} \quad \sigma(T|_{\mathcal{R}(E(\Delta'))}) \subseteq \Delta',$$

and hence, as $\Delta \subseteq \Lambda$ and $\Delta' \subseteq \Lambda' = \sigma(T)\backslash\Lambda$,

$$(**) \quad \sigma(T|_{\mathcal{R}(E(\Delta))}) \cap \sigma(T|_{\mathcal{R}(E(\Delta'))}) = \varnothing.$$

Therefore, according to Proposition 2.T, $(*)$ and $(**)$ imply that

$$E(\Delta')SE(\Delta) = O.$$

This identity holds for all $\Delta \in \mathcal{C}_\Lambda$ and all $\Delta' \in \mathcal{C}_{\Lambda'}$, where

$$\mathcal{C}_\Lambda = \big\{\Delta \in \mathcal{A}_{\sigma(T)}\colon \Delta \text{ is a closed subset of } \Lambda\big\}.$$

Now Proposition 3.C says that, in the inclusion ordering,

$$\mathcal{R}(E(\Lambda)) = \sup_{\Delta \in \mathcal{C}_\Lambda} \mathcal{R}(E(\Delta)) \quad \text{and} \quad \mathcal{R}(E(\Lambda')) = \sup_{\Delta' \in \mathcal{C}_{\Lambda'}} \mathcal{R}(E(\Delta'))$$

and so, in the extension ordering (see, e.g., [66, Problem 1.17]),

$$E(\Lambda) = \sup_{\Delta \in \mathcal{C}_\Lambda} E(\Delta) \quad \text{and} \quad E(\Lambda') = \sup_{\Delta' \in \mathcal{C}_{\Lambda'}} E(\Delta').$$

Then

$$E(\Lambda')SE(\Lambda) = O.$$

Since $E(\Lambda) + E(\Lambda') = E(\Lambda \cup \Lambda') = I$, this leads to

$$SE(\Lambda) = E(\Lambda \cup \Lambda')SE(\Lambda) = (E(\Lambda) + E(\Lambda'))SE(\Lambda) = E(\Lambda)SE(\Lambda).$$

Thus $\mathcal{R}(E(\Lambda))$ is S-invariant (cf. Proposition 1.I). Since this holds for every Λ in $\mathcal{A}_{\sigma(T)}$, it holds for $\Lambda' = \sigma(T)\backslash\Lambda$ so that $\mathcal{R}(E(\Lambda'))$ is S-invariant. But $\mathcal{R}(E(\Lambda)) \perp \mathcal{R}(E(\Lambda'))$ (i.e., $E(\Lambda)E(\Lambda') = O$) and $\mathcal{H} = \mathcal{R}(E(\Lambda)) + \mathcal{R}(E(\Lambda'))$ (since $\mathcal{H} = \mathcal{R}(E(\Lambda \cup \Lambda')) = \mathcal{R}(E(\Lambda) + E(\Lambda')) \subseteq \mathcal{R}(E(\Lambda)) + \mathcal{R}(E(\Lambda')) \subseteq \mathcal{H}$). Hence $\mathcal{R}(E(\Lambda')) = \mathcal{R}(E(\Lambda))^\perp$, and so $\mathcal{R}(E(\Lambda))^\perp$ also is S-invariant. Then

$$\mathcal{R}(E(\Lambda)) \text{ reduces } S,$$

which, by Proposition 1.I, is equivalent to

$$SE(\Lambda) = E(\Lambda)S. \qquad \square$$

   Invariant subspaces were discussed in Section 1.1 (a subspace $\mathcal{M}$ is T-invariant if $T(\mathcal{M}) \subseteq \mathcal{M}$). Recall that a subspace $\mathcal{M}$ of a normed space $\mathcal{X}$ is *nontrivial* if $\{0\} \neq \mathcal{M} \neq \mathcal{X}$. *Reducing subspaces* (i.e., subspaces $\mathcal{M}$ such that both $\mathcal{M}$ and $\mathcal{M}^\perp$ are T-invariant or, equivalently, a subspace $\mathcal{M}$ that is invariant for both T and T^*), and *reducible operators* (i.e., operators that have a nontrivial reducing subspace), were discussed in Section 1.5. *Hyperinvariant subspaces* were defined in Section 1.9 (subspaces that are invariant for every operator that commutes with a given operator). Recall that an operator is *nonscalar* if it is not a (complex) multiple of the identity — otherwise it is called a *scalar operator* (Section 1.7). Scalar operators are clearly normal. A projection E is *nontrivial* if $O \neq E \neq I$. Equivalently, *a projection is nontrivial if and only if it is nonscalar*. (Indeed, if $E^2 = E$, then it follows at once that $O \neq E \neq I$ if and only if $E \neq \alpha I$ for every $\alpha \in \mathbb{C}$).

Corollary 3.18. *Consider operators acting on a complex Hilbert space of dimension greater than* 1.

(a) *Every normal operator has a nontrivial reducing subspace.*

(b) *Every nonscalar normal operator has a nontrivial hyperinvariant subspace which reduces every operator that commutes with it.*

(c) *An operator is reducible if and only if it commutes with a nonscalar normal operator.*

(d) *An operator is reducible if and only if it commutes with a nontrivial orthogonal projection.*

(e) *An operator is reducible if and only if it commutes with a nonscalar operator that also commutes with its adjoint.*

Proof. Consider the spectral decomposition $T = \int \lambda \, dE_\lambda$ of a normal operator T as in Theorem 3.15. By Theorem 3.17, if $ST = TS$, then $SE(\Lambda) = E(\Lambda)S$ or, equivalently, each subspace $\mathcal{R}(E(\Lambda))$ reduces S, which means that $\{\mathcal{R}(E(\Lambda))\}$ is a family of reducing subspaces for every operator that commutes with T. If $\sigma(T)$ has a single point, say $\sigma(T) = \{\lambda\}$, then $T = \lambda I$ (by uniqueness of the spectral measure); that is, T is a scalar operator so that every subspace of $\mathcal{H}$ reduces T. If $\dim \mathcal{H} > 1$, then in this case T has plenty of *nontrivial* reducing subspaces. Hence, if T is nonscalar, then $\sigma(T)$ has more than one point (and $\dim \mathcal{H} > 1$). If λ, μ lie in $\sigma(T)$ and $\lambda \neq \mu$, then let

$$\mathbb{D}_\lambda = B_{\frac{1}{2}|\lambda - \mu|}(\lambda) = \left\{\nu \in \mathbb{C} \colon |\nu - \lambda| < \tfrac{1}{2}|\lambda - \mu|\right\}$$

be the open disk of radius $\frac{1}{2}|\lambda - \mu|$ centered at λ. Set

$$\Lambda_\lambda = \sigma(T) \cap \mathbb{D}_\lambda \quad \text{and} \quad \Lambda'_\lambda = \sigma(T) \backslash \mathbb{D}_\lambda,$$

so that $\sigma(T)$ is the disjoint union of Λ_λ and Λ'_λ. Observe that both Λ_λ and Λ'_λ lie in $\mathcal{A}_{\sigma(T)}$. Since Λ_λ and $\sigma(T) \backslash \mathbb{D}_\lambda^-$ are nonempty relatively open subsets of $\sigma(T)$ and $\sigma(T) \backslash \mathbb{D}_\lambda^- \subseteq \Lambda'_\lambda$, it follows by Theorem 3.15 that $E(\Lambda_\lambda) \neq O$ and $E(\Lambda'_\lambda) \neq O$. Then $I = E(\sigma(T)) = E(\Lambda_\lambda \cup \Lambda'_\lambda) = E(\Lambda_\lambda) + E(\Lambda'_\lambda)$, and so $E(\Lambda_\lambda) = I - E(\Lambda'_\lambda) \neq I$. Therefore,

$$O \neq E(\Lambda_\lambda) \neq I,$$

which means that $\mathcal{R}(E(\Lambda_\lambda))$ is nontrivial; that is,

$$\{0\} \neq \mathcal{R}(E(\Lambda_\lambda)) \neq \mathcal{H}.$$

This proves (a) (for the particular case of $S = T$), and also proves (b) — the nontrivial $\mathcal{R}(E(\Lambda_\lambda))$ reduces every S that commutes with T. Thus (b) ensures that if T commutes with a nonscalar normal, then it is reducible. The converse follows by Proposition 1.I, since every nontrivial orthogonal projection is a nonscalar normal. This proves (c). Since a projection E is nontrivial if and only if $\{0\} \neq \mathcal{R}(E) \neq \mathcal{H}$, Proposition 1.I ensures that T is reducible if and only if it commutes with a nontrivial orthogonal projection, which proves

(d). Since every nontrivial orthogonal projection is self-adjoint and nonscalar, it follows by (d) that if T is reducible then it commutes with a nonscalar operator that also commutes with its adjoint. Conversely, suppose T is such that $LT = TL$ and $LT^* = T^*L$ (and so $L^*T = TL^*$) for some nonscalar operator L. Then $(L + L^*)T = T(L + L^*)$, where $L + L^*$ is self-adjoint (thus normal). If this self-adjoint $L + L^*$ is not scalar, then T commutes with the nonscalar normal $L + L^*$. On the other hand, if this self-adjoint $L + L^*$ is scalar, then the nonscalar L must be normal. Indeed, if $L + L^* = \alpha I$, then $L^*L = LL^* = \alpha L - L^2$, which shows that L is normal. Again, in this case, T commutes with the nonscalar normal L. Thus, in any case, T commutes with a nonscalar normal operator so that T is reducible according to (c); and this completes the proof of (e). $\qquad\square$

All the equivalent assertions in Corollary 3.5, which hold for compact normal operators, remain equivalent in the general case (for plain normal, not necessarily compact operators). Again, the implication (a) $\Rightarrow$ (c) in Corollary 3.19 below is the Fuglede Theorem (Theorem 3.17).

Corollary 3.19. Let $T = \int \lambda\, dE_\lambda$ be a normal operator in $\mathcal{B}[\mathcal{H}]$ and take any operator S in $\mathcal{B}[\mathcal{H}]$. The following assertions are pairwise equivalent.

(a) *S commutes with T.*

(b) *S commutes with T^*.*

(c) *S commutes with every $E(\Lambda)$.*

(d) *$\mathcal{R}(E(\Lambda))$ reduces S for every Λ.*

Proof. Let $T = \int \lambda\, dE_\lambda$ be the spectral decomposition of a normal operator in $\mathcal{B}[\mathcal{H}]$. Take any $\Lambda \in \mathcal{A}_{\sigma(T)}$ and an arbitrary $S \in \mathcal{B}[\mathcal{H}]$. First we show that

$$ST = TS \quad\Longleftrightarrow\quad SE(\Lambda) = E(\Lambda)S \quad\Longleftrightarrow\quad ST^* = T^*S.$$

If $ST = TS$, then $SE(\Lambda) = E(\Lambda)S$ by Theorem 3.17 so that $\pi_{x,S^*y}(\Lambda) = \langle E(\Lambda)x\,;S^*y\rangle = \langle SE(\Lambda)x\,;y\rangle = \langle E(\Lambda)Sx\,;y\rangle = \pi_{Sx,y}(\Lambda)$, and hence

$$\langle ST^*x\,;y\rangle = \langle T^*x\,;S^*y\rangle = \int \overline{\lambda}\, d\langle E_\lambda x\,;S^*y\rangle$$

$$= \int \overline{\lambda}\, d\langle SE_\lambda x\,;y\rangle = \int \overline{\lambda}\, d\langle E_\lambda Sx\,;y\rangle = \langle T^*Sx\,;y\rangle,$$

for every $x, y \in \mathcal{H}$ (cf. Lemmas 3.7 and 3.8(a)). Thus

$$ST = TS \quad\Longrightarrow\quad SE(\Lambda) = E(\Lambda)S \quad\Longrightarrow\quad ST^* = T^*S \quad\Longleftrightarrow\quad S^*T = TS^*.$$

Since these hold for every $S \in \mathcal{B}[\mathcal{H}]$, and since $S^{**} = S$, it follows that

$$S^*T = TS^* \quad\Longrightarrow\quad ST = TS.$$

This shows that (a), (b), and (c) are equivalent, and (d) is equivalent to (c) by Proposition 1.I (as in the final part of the proof of Theorem 3.17).   □

Corollary 3.20. (FUGLEDE–PUTNAM THEOREM). *Suppose T_1 in $\mathcal{B}[\mathcal{H}]$ and T_2 in $\mathcal{B}[\mathcal{K}]$ are normal operators. If $X \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$ intertwines T_1 to T_2, then X intertwines T_1^* to T_2^* (i.e., if $XT_1 = T_2X$, then $XT_1^* = T_2^*X$).*

Proof. Take $T_1 \in \mathcal{B}[\mathcal{H}]$, $T_2 \in \mathcal{B}[\mathcal{K}]$, and $X \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$. Consider the operators

$$T = T_1 \oplus T_2 = \begin{pmatrix} T_1 & O \\ O & T_2 \end{pmatrix} \qquad \text{and} \qquad S = \begin{pmatrix} O & O \\ X & O \end{pmatrix}$$

in $\mathcal{B}[\mathcal{H} \oplus \mathcal{K}]$. If T_1 and T_2 are normal, then T is normal. If $XT_1 = T_2X$, then $ST = TS$ and so $ST^* = T^*S$ by Corollary 3.19. Hence $XT_1^* = T_2^*X$.   □

3.6 Additional Propositions

Let Γ be an arbitrary (not necessarily countable) nonempty index set. We have defined a diagonalizable operator in Section 3.2 as follows.

(1) An operator T on a Hilbert space $\mathcal{H}$ is *diagonalizable* if there is a resolution of the identity $\{E_\gamma\}_{\gamma \in \Gamma}$ on $\mathcal{H}$ and a bounded family of scalars $\{\lambda_\gamma\}_{\gamma \in \Gamma}$ such that $Tu = \lambda_\gamma u$ whenever $u \in \mathcal{R}(E_\gamma)$.

As we saw in Section 3.2, this can be equivalently stated as follows.

(1′) An operator T on a Hilbert space $\mathcal{H}$ is *diagonalizable* if it is a weighted sum of projections for some bounded sequence of scalars $\{\lambda_\gamma\}_{\gamma \in \Gamma}$ and some resolution of the identity $\{E_\gamma\}_{\gamma \in \Gamma}$ on $\mathcal{H}$.

Thus *every diagonalizable operator is normal* by Corollary 1.9, and what the Spectral Theorem for compact normal operators (Theorem 3.3) says is precisely that a *compact operator is normal if and only if it is diagonalizable* — cf. Corollary 3.5(a) — and so (being normal) a diagonalizable operator is such that $r(T) = \|T\| = \sup_{\gamma \in \Gamma} |\lambda_\gamma|$ (cf. Lemma 3.1). However, a diagonalizable operator on a separable Hilbert space was defined in Section 2.7 (Proposition 2.L), whose natural extension to arbitrary Hilbert space (not necessarily separable) in terms of summable families (see Section 1.3) rather than summable sequences, reads as follows.

(2) An operator T on a Hilbert space $\mathcal{H}$ is *diagonalizable* if there is an orthonormal basis $\{e_\gamma\}_{\gamma \in \Gamma}$ for $\mathcal{H}$ and a bounded family $\{\lambda_\gamma\}_{\gamma \in \Gamma}$ of scalars such that $Tx = \sum_{\gamma \in \Gamma} \lambda_\gamma \langle x \, ; e_\gamma \rangle e_\gamma$ for every $x \in \mathcal{H}$.

There is no ambiguity here; both definitions coincide: the resolution of the identity $\{E_\gamma\}_{\gamma \in \Gamma}$ behind the statement in (2) is given by $E_\gamma x = \langle x \, ; e_\gamma \rangle e_\gamma$ for every $x \in \mathcal{H}$ and every $\gamma \in \Gamma$ according to the Fourier series expansion $x = \sum_{\gamma \in \Gamma} \langle x \, ; e_\gamma \rangle e_\gamma$ of each $x \in \mathcal{H}$ in terms of orthonormal basis $\{e_\gamma\}_{\gamma \in \Gamma}$. Now

consider the Banach space ℓ_Γ^∞ made up of all bounded families $a = \{\alpha_\gamma\}_{\gamma \in \Gamma}$ of complex numbers and also the Hilbert space ℓ_Γ^2 consisting of all square-summable families $z = \{\zeta_\gamma\}_{\gamma \in \Gamma}$ of complex numbers indexed by Γ. The definition of a diagonal operator in $\mathcal{B}[\ell_\Gamma^2]$ goes as follows.

(3) A *diagonal operator* $\mathcal{B}[\ell_\Gamma^2]$ is a mapping $D: \ell_\Gamma^2 \to \ell_\Gamma^2$ such that $Dz = \{\alpha_\gamma \zeta_\gamma\}_{\gamma \in \Gamma}$ for every $z = \{\zeta_\gamma\}_{\gamma \in \Gamma} \in \ell_\Gamma^2$, where $a = \{\alpha_\gamma\}_{\gamma \in \Gamma} \in \ell_\Gamma^\infty$ is a (bounded) family of scalars. (Notation: $D = \mathrm{diag}\{\alpha_\gamma\}_{\gamma \in \Gamma}$.)

A diagonal operator D is clearly diagonalizable by the resolution of the identity $E_\gamma z = \langle z \, ; f_\gamma \rangle f_\gamma = \zeta_\gamma f_\gamma$ given by the *canonical orthonormal basis* $\{f_\gamma\}_{\gamma \in \Gamma}$ for ℓ_Γ^2, viz., $f_\gamma = \{\delta_{\gamma,\beta}\}_{\beta \in \Gamma}$ with $\delta_{\gamma,\beta} = 1$ if $\beta = \gamma$ and $\delta_{\gamma,\beta} = 0$ if $\beta \neq \gamma$. So D is normal with $r(D) = \|D\| = \|a\|_\infty = \sup_{\gamma \in \Gamma} |\alpha_\gamma|$. Consider the Fourier series expansion of an arbitrary x in $\mathcal{H}$ in terms of any orthonormal basis $\{e_\gamma\}_{\gamma \in \Gamma}$ for $\mathcal{H}$, namely, $x = \sum_{\gamma \in \Gamma} \langle x \, ; e_\gamma \rangle e_\gamma$, so that we can identify x in $\mathcal{H}$ with the square-summable family $\{\langle x \, ; e_\gamma \rangle\}_{\gamma \in \Gamma}$ in ℓ_Γ^2 (i.e., x is the image of $\{\langle x \, ; e_\gamma \rangle\}_{\gamma \in \Gamma}$ under a unitary transformation between the unitary equivalent spaces $\mathcal{H}$ and ℓ_Γ^2). Thus, if $Tx = \sum_{\gamma \in \Gamma} \lambda_\gamma \langle x \, ; e_\gamma \rangle e_\gamma$, then we can identify Tx with the square-summable family $\{\lambda_\gamma \langle x \, ; e_\gamma \rangle\}_{\gamma \in \Gamma}$ in ℓ_Γ^2, and so we can identify T with a diagonal operator D that takes $z = \{\langle x \, ; e_\gamma \rangle\}_{\gamma \in \Gamma}$ into $Dz = \{\lambda_\gamma \langle x \, ; e_\gamma \rangle\}_{\gamma \in \Gamma}$. This means that T and D are *unitarily equivalent*: there is a unitary transformation $U \in \mathcal{B}[\mathcal{H}, \ell_\Gamma^2]$ such that $UT = DU$. It is in this sense that T is said to be a *diagonal operator with respect to the orthonormal basis* $\{e_\gamma\}_{\gamma \in \Gamma}$. Thus definition (2) can be rephrased as follows.

(2') An operator T on a Hilbert space $\mathcal{H}$ is *diagonalizable* if it is a diagonal operator with respect to some orthonormal basis for $\mathcal{H}$.

Proposition 3.A. *Let T be an operator on a nonzero complex Hilbert space. The following assertions are equivalent.*

(a) *T is diagonalizable in the sense of definition (1) above.*

(b) *$\mathcal{H}$ has an orthogonal basis made up of eigenvectors of T.*

(c) *T is diagonalizable in the sense of definition (2) above.*

(d) *T is unitarily equivalent to a diagonal operator.*

Take a measure space $(\Omega, \mathcal{A}_\Omega, \mu)$ where $\mathcal{A}_\Omega$ is a σ-algebra of subsets of a nonempty set Ω and μ is a positive measure on $\mathcal{A}_\Omega$. Let $L^\infty(\Omega, \mathbb{C}, \mu)$ — short notation: $L^\infty(\Omega, \mu)$, $L^\infty(\mu)$, or just L^∞ — denote the Banach space of all *essentially bounded $\mathcal{A}_\Omega$-measurable complex-valued functions* $f: \Omega \to \mathbb{C}$ on Ω equipped with the usual sup-norm (i.e., $\|f\|_\infty = \mathrm{ess\,sup}|f|$ for $f \in L^\infty$). Let $L^2(\Omega, \mathbb{C}, \mu)$ — short notation: $L^2(\Omega, \mu)$, $L^2(\mu)$, or just L^2 — denote the Hilbert space of all *square-integrable $\mathcal{A}_\Omega$-measurable complex-valued functions* $f: \Omega \to \mathbb{C}$ on Ω equipped with the usual L^2-norm (i.e., $\|f\|_2 = \left(\int |f|^2 d\mu\right)^{\frac{1}{2}}$ for $f \in L^2$). *Multiplication operators* on L^2 will be considered next (see, e.g.,

[50, Chapter 7]). The first version of the Spectral Theorem (Theorem 3.11) says that multiplication operators are prototypes of normal operators.

Proposition 3.B. *Let* $\phi\colon \Omega \to \mathbb{C}$ *and* $f\colon \Omega \to \mathbb{C}$ *be complex-valued functions on a nonempty set* Ω. *Let* μ *be a positive measure on a* σ-*algebra* $\mathcal{A}_\Omega$ *of subsets of* Ω. *Suppose* $\phi \in L^\infty$. *If* $f \in L^2$, *then* $\phi f \in L^2$. *Thus consider the mapping* $M_\phi\colon L^2 \to L^2$ *defined by*

$$M_\phi f = \phi f \quad \text{for every} \quad f \in L^2.$$

That is, $(M_\phi f)(\lambda) = \phi(\lambda)f(\lambda)$ *for* $\lambda \in \Omega$. *This is called the* multiplication operator *on* L^2, *which is linear and bounded (i.e.,* $M_\phi \in \mathcal{B}[L^2]$). *Moreover:*

(a) $M_\phi^* = M_{\overline{\phi}}$ (*i.e.,* $(M_\phi^* f)(\lambda) = \overline{\phi(\lambda)}\, f(\lambda)$ *for* $\lambda \in \Omega$).

(b) M_ϕ *is a normal operator*.

(c) $\|M_\phi\| \le \|\phi\|_\infty = \operatorname{ess\,sup} |\phi| = \inf_{\Lambda \in \mathcal{N}_\Omega} \sup_{\lambda \in \Omega\backslash\Lambda} |\phi(\lambda)|,$

   where $\mathcal{N}_\Omega$ *denotes the collection of all* $\Lambda \in \mathcal{A}_\Omega$ *such that* $\mu(\Lambda) = 0$.

(d) $\sigma(M_\phi) \subseteq \operatorname{ess} \mathcal{R}(\phi) = $ *essential range of* ϕ

$$= \big\{\alpha \in \mathbb{C}\colon \mu(\{\lambda \in \Omega\colon |\phi(\lambda) - \alpha| < \varepsilon\}) > 0 \text{ for every } \varepsilon > 0\big\}$$

$$= \big\{\alpha \in \mathbb{C}\colon \mu(\phi^{-1}(V_\alpha)) > 0 \text{ for every neighborhood } V_\alpha \text{ of } \alpha\big\}$$

$$= \bigcap \big\{\phi(\Lambda)^- \in \mathbb{C}\colon \Lambda \in \mathcal{A}_\Omega \text{ and } \mu(\Omega\backslash\Lambda) = 0\big\} \subseteq \phi(\Omega)^- = \mathcal{R}(\phi)^-.$$

If, in addition, μ *is* σ-*finite, then*

(e) $\|M_\phi\| = \|\phi\|_\infty$,

(f) $\sigma(M_\phi) = \operatorname{ess} \mathcal{R}(\phi)$.

   Particular case: *Suppose* Ω *is a nonempty bounded subset of* $\mathbb{C}$ *and suppose* $\varphi\colon \Omega \to \Omega \subseteq \mathbb{C}$ *is the identity function on* Ω *(i.e.,* $\varphi(\lambda) = \lambda$ *for every* $\lambda \in \Omega$, *which is bounded). Then* $\operatorname{ess} \mathcal{R}(\varphi) = \bigcap\{\Lambda^- \in \mathbb{C}\colon \Lambda \in \mathcal{A}_\Omega \text{ and } \mu(\Omega\backslash\Lambda) = 0\} \subseteq \Omega^-$, *and so* $\operatorname{ess} \mathcal{R}(\varphi) \cap \Omega$ *is the smallest relatively closed subset of* Ω *such that* $\mu(\Omega\backslash(\operatorname{ess} \mathcal{R}(\varphi) \cap \Omega)) = 0$. *Equivalently,* $\Omega\backslash(\operatorname{ess} \mathcal{R}(\varphi) \cap \Omega)$ *is the largest relatively open subset of* Ω *of measure zero. Thus* $\operatorname{ess} \mathcal{R}(\varphi) \cap \Omega$ *is the (bounded and relatively closed)* support *of* μ. *Summing up, if* Ω *is a nonempty bounded subset of* $\mathbb{C}$, *and* $\varphi\colon \Omega \to \Omega$ *is the identity function on* Ω, *then* support$(\mu) = \operatorname{ess} \mathcal{R}(\varphi) \cap \Omega$. *If, in addition,* Ω *is closed in* $\mathbb{C}$, *then* support(μ) *is compact. Therefore, if* Λ *in* $\mathcal{A}_\Omega$ *is nonempty and relatively open, included in the support of* μ, *then* $\mu(\Lambda) > 0$. *(Indeed, if* $\varnothing \ne \Lambda \subseteq \operatorname{ess} \mathcal{R}(\varphi) \cap \Omega$ *is relatively open and* $\mu(\Lambda) = 0$, *then* $[\Omega\backslash(\operatorname{ess} \mathcal{R}(\varphi) \cap \Omega)] \cup \Lambda$ *is a relatively open subset of* Ω *larger than* $\Omega\backslash(\operatorname{ess} \mathcal{R}(\varphi) \cap \Omega)$ *and of measure zero; which is a contradiction.)*

   If $\Omega = \mathbb{N}$, the set of all positive integers, $\mathcal{A}_\Omega = \wp(\mathbb{N})$, the power set of $\mathbb{N}$, and μ is the counting measure (assigning to each set the number of elements

in it, which is σ-finite), then a multiplication operator is reduced to a diagonal operator on $L^2(\mathbb{N}, \mathbb{C}, \mu) = \ell_{\mathbb{N}}^2 = \ell_+^2$, where the bounded $\wp(\mathbb{N})$-measurable function $\phi: \mathbb{N} \to \mathbb{C}$ in $L^\infty(\mathbb{N}, \mathbb{C}, \mu) = \ell_{\mathbb{N}}^\infty = \ell_+^\infty$ is the bounded sequence $\{\alpha_k\}$ with $\alpha_k = \phi(k)$ for each $k \in \mathbb{N}$, so that $\|M_\phi\| = \|\phi\|_\infty = \sup_k |\alpha_k|$ and $\sigma(\mathcal{M}_\phi) = \mathcal{R}(\phi)^- = \{\alpha_k\}^-$. What the Spectral Theorem says is that a normal operator is unitarily equivalent to a multiplication operator (Theorem 3.11), which turns out to be a diagonal operator in the compact case (Theorem 3.3).

Proposition 3.C. *Let Ω be a nonempty compact subset of the complex plane $\mathbb{C}$ and let $E: \mathcal{A}_\Omega \to \mathcal{B}[\mathcal{H}]$ be a spectral measure (cf. Definition 3.6). For each $\Lambda \in \mathcal{A}_\Omega$ set $\mathcal{C}_\Lambda = \{\Delta \in \mathcal{A}_\Omega \colon \Delta \text{ is a closed subset of } \Lambda\}$. $\mathcal{R}(E(\Lambda))$ is the smallest subspace of $\mathcal{H}$ such that $\bigcup_{\Delta \in \mathcal{C}_\Lambda} \mathcal{R}(E(\Delta)) \subseteq \mathcal{R}(E(\Lambda))$.*

Proposition 3.D. *Let $\mathcal{H}$ be a complex Hilbert space and let $\mathbb{T}$ be the unit circle about the origin of the complex plane. If $T \in \mathcal{B}[\mathcal{H}]$ is normal, then*

(a) *T is unitary if and only if $\sigma(T) \subseteq \mathbb{T}$,*

(b) *T is self-adjoint if and only if $\sigma(T) \subseteq \mathbb{R}$,*

(c) *T is nonnegative if and only if $\sigma(T) \subseteq [0, \infty)$,*

(d) *T is strictly positive if and only if $\sigma(T) \subseteq [\alpha, \infty)$ for some $\alpha > 0$,*

(e) *T is an orthogonal projection if and only if $\sigma(T) \subseteq \{0, 1\}$.*

Observe that, if $T = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ in $\mathcal{B}[\mathbb{C}^2]$, then $S^2 - T^2 = \left(\begin{smallmatrix} 4 & 3 \\ 3 & 2 \end{smallmatrix}\right)$. Thus $O \leq T \leq S$ does not imply $T^2 \leq S^2$. However, the converse holds.

Proposition 3.E. *If $T, S \in \mathcal{B}[\mathcal{H}]$ are nonnegative operators, then*

$$T \leq S \quad \text{implies} \quad T^{\frac{1}{2}} \leq S^{\frac{1}{2}}.$$

Proposition 3.F. *If $T = \int \lambda \, dE_\lambda$ is the spectral decomposition of a nonnegative operator T on a complex Hilbert space, then $T^{\frac{1}{2}} = \int \lambda^{\frac{1}{2}} \, dE_\lambda$.*

Proposition 3.G. *If $T = \int \lambda \, dE_\lambda$ is the spectral decomposition of a normal operator $T \in \mathcal{B}[\mathcal{H}]$, and if λ_0 is an isolated point of $\sigma(T)$, then*

$$\{0\} \neq \mathcal{R}(E(\{\lambda_0\})) \subseteq \mathcal{N}(\lambda_0 I - T).$$

Every isolated point of the spectrum of a normal operator is an eigenvalue.

Recall: $X \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$ *intertwines* $T \in \mathcal{B}[\mathcal{H}]$ to $S \in \mathcal{B}[\mathcal{K}]$ if $XT = SX$. If an invertible X intertwines T to S, then T and S are *similar*. If, in addition, X is unitary, then T and S are *unitarily equivalent* (cf. Section 1.9).

Proposition 3.H. *Let $T_1 \in \mathcal{B}[\mathcal{H}]$ and $T_2 \in \mathcal{B}[\mathcal{K}]$ be normal operators. If $X \in \mathcal{B}[\mathcal{H}, \mathcal{K}]$ intertwines T_1 to T_2 (i.e., $X T_1 = T_2 X$), then*

(a) $\mathcal{N}(X)$ *reduces* T_1 *and* $\mathcal{R}(X)^-$ *reduces* T_2

so that $T_1|_{\mathcal{N}(X)^\perp} \in \mathcal{B}[\mathcal{N}(X)^\perp]$ *and* $T_2|_{\mathcal{R}(X)^-} \in \mathcal{B}[\mathcal{R}(X)^-]$. *Moreover,*

(b) $T_1|_{\mathcal{N}(X)^\perp}$ *and* $T_2|_{\mathcal{R}(X)^-}$ *are unitarily equivalent.*

A special case of Proposition 3.H says that *if a quasiinvertible* (i.e., injective with dense range) *bounded linear transformation intertwines two normal operators, then these normal operators are unitarily equivalent.* This happens, in particular, when X is invertible (i.e., if X is in $\mathcal{G}[\mathcal{H}, \mathcal{K}]$).

Proposition 3.I. *Two similar normal operators are unitarily equivalent.*

A subset of a metric space is *nowhere dense* if its closure has empty interior. Since the spectrum of $T \in \mathcal{B}[\mathcal{H}]$ is closed, $\sigma(T)$ is nowhere dense if and only if it has an empty interior; that is, $\sigma(T)^\circ = \varnothing$. The full spectrum $\sigma(T)^\#$ was defined in Section 2.7 (see Proposition 2.S). Note that the condition $\sigma(T) = \sigma(T)^\#$ is equivalent to saying that $\mathbb{C}\backslash\sigma(T)$ is connected (i.e., that $\rho(T)$ is connected). An operator $T \in \mathcal{B}[\mathcal{H}]$ is *reductive* if all its invariant subspaces are reducing. Thus, by Proposition 1.Q, *a normal operator is reductive if and only if its restriction to every invariant subspace is normal.* Normal reductive operators are also called *completely normal.*

Proposition 3.J. *Suppose* $T \in \mathcal{B}[\mathcal{H}]$ *is a normal operator. If* $\sigma(T) = \sigma(T)^\#$ *and* $\sigma(T)^\circ = \varnothing$, *then* T *is reductive. If* T *is reductive, then* $\sigma(T)^\circ = \varnothing$.

There are normal reductive operators D for which $\sigma(D) \neq \sigma(D)^\#$. For instance, consider the diagonal $D = \text{diag}\{\lambda_k\}_{k\geq 0}$ in $\mathcal{B}[\ell_+^2]$ with $\lambda_k = e^{2\pi i \alpha_k}$, where $\{\alpha_k\}_{k\geq 0}$ is a distinct enumeration of all rationals in $[0,1)$. Being a diagonal, D is normal (unitary, actually) and $\sigma(D) = \mathbb{T}$ (since $\{\alpha_k\}_{k\geq 0}$ is dense in $[0,1]$). Thus $\mathbb{T} = \sigma(D) \subset \sigma(D)^\# = \mathbb{D}^-$. However, D is reductive (see, e.g., [35, Example 13.5]). On the other hand, there are normal operators S with $\sigma(S)^\circ = \varnothing$ that are not reductive. Indeed, every bilateral shift S has an invariant subspace $\mathcal{M}$ such that the restriction $S_+ = S|_{\mathcal{M}}$ is a unilateral shift (see, e.g., [62, Proposition 2.13]). Since S is unitary (thus normal) and S_+ is not normal, it follows by Proposition 1.Q that $\mathcal{M}$ does not reduce S. Thus S is not reductive, although $\sigma(S) = \mathbb{T}$, and so $\sigma(S)^\circ = \varnothing$.

Proposition 3.K. *Let* T *be a diagonalizable operator on a nonzero complex Hilbert space. The following assertions are equivalent.*

(a) *Every nontrivial invariant subspace for* T *contains an eigenvector.*

(b) *Every nontrivial invariant subspace* $\mathcal{M}$ *for* T *is spanned by the eigenvectors of* $T|_{\mathcal{M}}$ *(i.e., it is such that* $\mathcal{M} = \bigvee_{\lambda \in \sigma_P(T|_{\mathcal{M}})} \mathcal{N}(\lambda I - T|_{\mathcal{M}})$).

(c) T *is reductive.*

There are diagonalizable operators that are not reductive. In fact, since $\mathbb{C}$ is a separable metric space, it includes a countable dense subset, and so

does every compact subset of $\mathbb{C}$. Let Λ be any countable dense subset of an arbitrary nonempty compact subset Ω of $\mathbb{C}$. Let $\{\lambda_k\}$ be an enumeration of Λ so that $\sup_k |\lambda_k| < \infty$ (since Ω is bounded). Consider the diagonal operator $D = \mathrm{diag}\{\lambda_k\}$ in $\mathcal{B}[\ell_+^2]$. As we had seen in Proposition 2.L,

$$\sigma(D) = \Lambda^- = \Omega.$$

Therefore, *every closed and bounded subset of the complex plane is the spectrum of some diagonal operator on ℓ_+^2.* Hence, if $\Omega^\circ \neq \varnothing$, then Proposition 3.J ensures that the diagonal D is not reductive.

Notes: A discussion on diagonalizable operators precedes the characterization in Proposition 3.A (see, e.g., [66, Problem 6.25]). Diagonal operators are extended to multiplication operators, whose properties that will be needed in the text are summarized in Proposition 3.B (see, e.g., [27, IX.2.6], [50, Chapter 7], and [76, p. 13]). For the technical result in Proposition 3.C, which is required in the proof of the Fuglede Theorem (Theorem 3.17), see [76, Proposition 1.4]. The remaining propositions depend on the Spectral Theorem (Theorem 3.15). Proposition 3.D considers the converse of Proposition 2.A (see, e.g., [66, Problem 6.26]). For Proposition 3.E see, for instance, [63, Problem 8.12]. Proposition 3.F is a consequence of Lemma 3.10 and Proposition 1.M (see, e.g., [66, Problem 6.45]). Proposition 3.G is an important classical result that shows how the spectral decomposition of a normal operator behaves similarly to the compact case of Theorem 3.3 for isolated points of the spectrum (compare with the claim in the proof of Lemma 3.2) — see, e.g., [66, Problem 6.28]. Propositions 3.H and 3.I follow from Corollary 3.20 (see, e.g., [27, IX.6.10 and IX.6.11]). Reductive normal operators are considered in Propositions 3.J and 3.K (see [35, Theorems 13.2 and 13.32] and [76, Theorems 1.23 and 1.25]).

Suggested Reading

Arveson [6]

Bachman and Narici [8]

Berberian [14]

Conway [27, 29]

Douglas [34]

Dowson [35]

Dunford and Schwartz [40]

Halmos [47, 50]

Helmberg [54]

Kubrusly [66]

Radjavi and Rosenthal [76]

Rudin [80]

4

Functional Calculus

Fix an operator T in the operator algebra $\mathcal{B}[\mathcal{H}]$. Suppose an operator $\psi(T)$ in $\mathcal{B}[\mathcal{H}]$ can be associated to each function $\psi\colon \Omega \to \mathbb{C}$ on a nonempty set Ω, in a suitable algebra of functions $F(\Omega)$. The central theme of this chapter investigates the mapping $\Phi_T\colon F(\Omega) \to \mathcal{B}[\mathcal{H}]$ defined by $\Phi_T(\psi) = \psi(T)$ for every ψ in $F(\Omega)$; that is, the mapping $\psi \mapsto \psi(T)$ that takes each function ψ in a function algebra $F(\Omega)$ to the operator $\psi(T)$ in the operator algebra $\mathcal{B}[\mathcal{H}]$. If this mapping Φ_T is linear and also preserves product (i.e., if it is an *algebra homomorphism*), then it is referred to as a *functional calculus* for T.

4.1 Rudiments of C*-Algebra

Let $\mathcal{A}$ and $\mathcal{B}$ be algebras over the same scalar field $\mathbb{F}$. If $\mathbb{F} = \mathbb{C}$, then these are referred to as *complex algebras*. A linear transformation $\Phi\colon \mathcal{A} \to \mathcal{B}$ (of the linear space $\mathcal{A}$ into the linear space $\mathcal{B}$) that preserves products (i.e., such that $\Phi(xy) = \Phi(x)\Phi(y)$ for every x, y in $\mathcal{A}$) is a *homomorphism* (or an *algebra homomorphism*) of $\mathcal{A}$ into $\mathcal{B}$. A *unital algebra* (or an *algebra with identity*) is an algebra with an identity element (i.e., with a neutral element under multiplication). An element x in a unital algebra $\mathcal{A}$ is *invertible* if there is an $x^{-1} \in \mathcal{A}$ such that $x^{-1}x = xx^{-1} = 1$, where 1 denotes the identity in $\mathcal{A}$. A *unital homomorphism* between unital algebras is one that takes the identity of $\mathcal{A}$ to the identity of $\mathcal{B}$. If Φ is an isomorphism (of the linear spaces $\mathcal{A}$ *onto* the linear space $\mathcal{B}$) and also a homomorphism (of the algebra $\mathcal{A}$ *onto* the algebra $\mathcal{B}$), then it is an *algebra isomorphism of* $\mathcal{A}$ *onto* $\mathcal{B}$. In this case $\mathcal{A}$ and $\mathcal{B}$ are said to be *isomorphic algebras*. A *normed algebra* is an algebra $\mathcal{A}$ that is also a normed space whose norm satisfies the operator norm property, viz., $\|xy\| \leq \|x\|\,\|y\|$ for every x, y in $\mathcal{A}$. The identity element of a unital normed algebra has norm 1. A *Banach algebra* is a complete normed algebra. The *spectrum* $\sigma(x)$ of an element $x \in \mathcal{A}$ in a *unital complex Banach algebra* $\mathcal{A}$ is the complement of the set $\rho(x) = \{\lambda \in \mathbb{C}\colon \lambda 1 - x \text{ has an inverse in } \mathcal{A}\}$, and its *spectral radius* is the number $r(x) = \sup_{\lambda \in \sigma(x)} |\lambda|$. An *involution* $*\colon \mathcal{A} \to \mathcal{A}$ on an algebra $\mathcal{A}$ is a map $x \mapsto x^*$ such that $(x^*)^* = x$, $(xy)^* = y^*x^*$, and $(\alpha x + \beta y)^* = \overline{\alpha}x^* + \overline{\beta}y^*$ for all scalars α, β and every x, y in $\mathcal{A}$. A *-algebra*

(or an *involutive algebra*) is an algebra equipped with an involution. If $\mathcal{A}$ and $\mathcal{B}$ are ∗-algebras, then a ∗-*homomorphism* (a ∗-*isomorphism*) between them is an algebra homomorphism $\Phi\colon \mathcal{A} \to \mathcal{B}$ (an algebra isomorphism) that preserves the involution (i.e., $\Phi(x^*) = \Phi(x)^*$ in $\mathcal{B}$ for every x in $\mathcal{A}$ — we are using the same notation for the involutions on $\mathcal{A}$ and $\mathcal{B}$). An element x in a ∗-algebra is called *Hermitian* if $x^* = x$ and *normal* if $x^*x = xx^*$. An element x in a unital ∗-algebra is *unitary* if $x^*x = xx^* = 1$. A C*-*algebra* is a Banach ∗-algebra $\mathcal{A}$ such that $\|x^*x\| = \|x\|^2$ for every x in $\mathcal{A}$.

In a ∗-algebra, $(x^*)^*$ is usually denoted by x^{**}. The origin 0 in a ∗-algebra and the identity 1 in a unital ∗-algebra are Hermitian. Indeed, since $(\alpha x)^* = \overline{\alpha}x^*$ for every x and every scalar α, we get $0^* = 0$ by setting $\alpha = 0$. Since $1^*x = (1^*x)^{**} = (x^*1)^* = (1x^*)^* = x1^*$ and $(x^*1)^* = x^{**} = x$, it follows that $1^*x = x1^* = x$, and so (by uniqueness of the identity) $1^* = 1$. In a unital C*-algebra the expression $\|1\| = 1$ is a theorem (rather than an axiom) since $\|1\|^2 = \|1^*1\| = \|1^2\| = \|1\|$ so that $\|1\| = 1$ because $1 \neq 0$.

Elementary properties of Banach algebras (in particular, C*-algebras) and of algebra homomorphisms that will be needed in the sequel are brought together in the following lemma.

Lemma 4.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be algebras over the same scalar field and let $\Phi\colon \mathcal{A} \to \mathcal{B}$ be a homomorphism. Take an arbitrary $x \in \mathcal{A}$.*

(a) *If $\mathcal{A}$ is a ∗-algebra, then $x = a + ib$ where $a^* = a$ and $b^* = b$ in $\mathcal{A}$.*

(b) *If $\mathcal{A}$, $\mathcal{B}$, and Φ are unital and x is invertible, then $\Phi(x)^{-1} = \Phi(x^{-1})$.*

(c) *If $\mathcal{A}$ is unital and normed, and x is invertible, then $\|x\|^{-1} \leq \|x^{-1}\|$.*

(d) *If $\mathcal{A}$ is a unital ∗-algebra and x is invertible, then $(x^{-1})^* = (x^*)^{-1}$.*

(e) *If $\mathcal{A}$ is a C*-algebra and $x^*x = 1$, then $\|x\| = 1$.*

(f) *If $\mathcal{A}$ is a C*-algebra, then $\|x^*\| = \|x\|$.*

Suppose $\mathcal{A}$ and $\mathcal{B}$ are unital complex Banach algebras.

(g) *If Φ is a unital homomorphism, then $\sigma(\Phi(x)) \subseteq \sigma(x)$.*

(h) *If Φ is an injective unital homomorphism, then $\sigma(\Phi(x)) = \sigma(x)$.*

(i) $r(x) = \lim_n \|x^n\|^{\frac{1}{n}}$.

(j) *If $\mathcal{A}$ is a unital complex C*-algebra and $x^* = x$, then $r(x) = \|x\|$.*

From now on assume that $\mathcal{A}$ and $\mathcal{B}$ are unital complex C-algebras.*

(k) *If Φ is a unital ∗-homomorphism, then $\|\Phi(x)\| \leq \|x\|$.*

(ℓ) *If Φ is an injective unital ∗-homomorphism, then $\|\Phi(x)\| = \|x\|$.*

Proof. Let $\Phi\colon \mathcal{A} \to \mathcal{B}$ be a homomorphism and take an arbitrary $x \in \mathcal{A}$.

(a) This is the *Cartesian decomposition* (as in Proposition 1.O). If $\mathcal{A}$ is a $*$-algebra, then set $a = \frac{1}{2}(x^* + x) \in \mathcal{A}$ and $b = \frac{i}{2}(x^* - x) \in \mathcal{A}$ so that $a^* = \frac{1}{2}(x + x^*) = a$, $b^* = \frac{-i}{2}(x - x^*) = b$, and $a + ib = \frac{1}{2}(x^* + x - x^* + x) = x$.

(b) If $\mathcal{A}$ and $\mathcal{B}$ are unital algebras, Φ a unital homomorphism, and x invertible, then $1 = \Phi(1) = \Phi(x^{-1}x) = \Phi(xx^{-1}) = \Phi(x^{-1})\Phi(x) = \Phi(x)\Phi(x^{-1})$.

(c) If $\mathcal{A}$ is a unital normed algebra, then $1 = \|1\| = \|x^{-1}x\| \leq \|x^{-1}\|\,\|x\|$ whenever x is invertible.

(d) If $\mathcal{A}$ is a unital $*$-algebra and if $x^{-1}x = xx^{-1} = 1$, then $x^*(x^{-1})^* = (x^{-1}x)^* = (xx^{-1})^* = (x^{-1})^*x^* = 1^* = 1$, and so $(x^*)^{-1} = (x^{-1})^*$.

(e) If $\mathcal{A}$ is a C*-algebra and if $x^*x = 1$, then $\|x\|^2 = \|x^*x\| = \|1\| = 1$.

(f) Let $\mathcal{A}$ be a C*-algebra. The result is trivial for $x = 0$ because $0^* = 0$. Since $\|x\|^2 = \|x^*x\| \leq \|x^*\|\,\|x\|$, it follows that $\|x\| \leq \|x^*\|$ if $x \neq 0$. Replacing x with x^* and recalling that $x^{**} = x$ we get $\|x^*\| \leq \|x\|$.

(g) Let $\mathcal{A}$ and $\mathcal{B}$ be unital complex Banach algebras, and let Φ be a unital homomorphism. If $\lambda \in \rho(x)$, then $\lambda 1 - x$ is invertible in $\mathcal{A}$. Since Φ is unital, $\Phi(\lambda 1 - x) = \lambda 1 - \Phi(x)$ is invertible in $\mathcal{B}$ by (b). Hence $\lambda \in \rho(\Phi(x))$. Thus $\rho(x) \subseteq \rho(\Phi(x))$. Therefore, $\mathbb{C} \backslash \rho(\Phi(x)) \subseteq \mathbb{C} \backslash \rho(x)$.

(h) If $\Phi: \mathcal{A} \to \mathcal{B}$ is injective, then it has an inverse $\Phi^{-1}: \mathcal{R}(\Phi) \to \mathcal{B}$ on its range $\mathcal{R}(\Phi) = \Phi(\mathcal{A}) \subseteq \mathcal{B}$ which, being the image of a unital algebra under a unital homomorphism, is again a unital algebra. Moreover, Φ^{-1} itself is a unital homomorphism. Indeed, if $y = \Phi(u)$ and $z = \Phi(v)$ are arbitrary elements in $\mathcal{R}(\Phi)$, then $\Phi^{-1}(yz) = \Phi^{-1}(\Phi(u)\Phi(v)) = \Phi^{-1}(\Phi(uv)) = uv = \Phi^{-1}(y)\Phi^{-1}(z)$, and so Φ^{-1} is a homomorphism (since inverses of linear transformations are linear) which is trivially unital (since Φ is unital). Now apply (b) so that $\Phi^{-1}(y)$ is invertible in $\mathcal{A}$ whenever y is invertible in $\mathcal{R}(\Phi) \subseteq \mathcal{B}$. If $\lambda \in \rho(\Phi(x))$, then $y = \lambda 1 - \Phi(x) = \Phi(\lambda 1 - x)$ is invertible in $\mathcal{R}(\Phi)$, and therefore $\Phi^{-1}(y) = \Phi^{-1}(\Phi(\lambda 1 - x)) = \lambda 1 - x$ is invertible in $\mathcal{A}$, which means that $\lambda \in \rho(x)$. Thus $\rho(\Phi(x)) \subseteq \rho(x)$. Hence $\rho(x) = \rho(\Phi(x))$ by (g).

(i) The proof of the *Gelfand–Beurling formula* in the complex Banach algebra $\mathcal{B}[\mathcal{X}]$ of Theorem 2.10 holds in any unital complex Banach algebra.

(j) If $\mathcal{A}$ is a C*-algebra and if $x^* = x$, then $\|x\|^2 = \|x^*x\| = \|x^2\|$ and so, by induction, $\|x\|^{2n} = \|x^{2n}\|$ for every integer $n \geq 1$. If $\mathcal{A}$ is unital and complex, then $r(x) = \lim_n \|x^n\|^{\frac{1}{n}} = \lim_n \|x^{2n}\|^{\frac{1}{2n}} = \lim_n (\|x\|^{2n})^{\frac{1}{2n}} = \|x\|$ by (i).

(k) Let $\mathcal{A}$ and $\mathcal{B}$ be complex unital C*-algebras and Φ a $*$-homomorphism. Then $\|\Phi(x)\|^2 = \|\Phi(x)^*\Phi(x)\| = \|\Phi(x^*)\Phi(x)\| = \|\Phi(x^*x)\|$. Observe that x^*x is Hermitian. Thus $\Phi(x^*x)$ is Hermitian since Φ is a $*$-homomorphism. Hence $\|\Phi(x^*x)\| = r(\Phi(x^*x)) \leq r(x^*x) = \|x^*x\| = \|x\|^2$ by (j) and (g).

(ℓ) Consider the setup and the argument of the preceding proof and suppose, in addition, that the unital $*$-homomorphism $\Phi: \mathcal{A} \to \mathcal{B}$ is injective. Thus apply (h) instead of (g) to get (ℓ) instead of (k). $\square$

Remark. Lemma 4.1(k) says that a unital ∗-homomorphism Φ between unital complex C*-algebras is a contraction, and so it is a contractive unital ∗-homomorphism. If it is injective, then Lemma 4.1(ℓ) says that Φ is an isometry, and so an isometric unital ∗-homomorphism. Since an isometry of a Banach space into a normed space has a closed range (see, e.g., [66, Problem 4.41]), it follows that, if a unital ∗-homomorphism Φ between unital complex C*-algebras $\mathcal{A}$ and $\mathcal{B}$ is injective, then the unital ∗-algebra $\mathcal{R}(\Phi) = \Phi(\mathcal{A}) \subseteq \mathcal{B}$, being closed in the Banach space $\mathcal{B}$, is itself a Banach space, and so $\mathcal{R}(\Phi)$ is a unital complex C*-algebra. Thus the injective unital ∗-homomorphism $\Phi: \mathcal{A} \to \mathcal{B}$ is such that $\Phi: \mathcal{A} \to \mathcal{R}(\Phi)$ is an *isometric isomorphism* (i.e., an injective and surjective linear isometry), and therefore $\mathcal{A}$ and $\mathcal{R}(\Phi)$ are *isometrically isomorphic* unital complex C*-algebras.

These are the elementary results on C*-algebras that will be needed in this chapter. For a thorough treatment of C*-algebras the reader is referred, for instance, to [5], [30], [43], and [74].

Throughout this chapter $\mathcal{H}$ stands for a *nonzero complex* Hilbert space, so that $\mathcal{B}[\mathcal{H}]$ is a (unital complex) C*-algebra, where involution is defined as the adjoint operation. Take an arbitrary nonempty set Ω and let $F(\Omega) = \mathbb{C}^{\Omega}$ be an algebra of complex-valued functions on Ω, where addition, scalar multiplication, and product are pointwise defined as usual. If $F(\Omega)$ is such that $\psi^* \in F(\Omega)$ whenever $\psi \in F(\Omega)$, where ψ^* is given by $\psi^*(\lambda) = \overline{\psi(\lambda)}$ for every $\lambda \in \Omega$, then complex conjugation defines an involution on $F(\Omega)$. In this case $F(\Omega)$ is a commutative ∗-algebra. If, in addition, $F(\Omega)$ is endowed with a norm that makes it complete (and so a Banach space), and if $\|\psi^*\psi\| = \|\psi\|^2$ (which is the case of the sup-norm $\|\cdot\|_{\infty}$ since $(\psi^*\psi)(\lambda) = \psi^*(\lambda)\psi(\lambda) = \overline{\psi(\lambda)}\psi(\lambda) = |\psi(\lambda)|^2$ for every $\lambda \in \Omega$), then $F(\Omega)$ is a (complex) C*-algebra.

Take any operator T in $\mathcal{B}[\mathcal{H}]$. The first steps towards a *functional calculus* for T are given by polynomials of T. To each polynomial $p: \sigma(T) \to \mathbb{C}$ with complex coefficients, $p(\lambda) = \sum_{i=0}^{n} \alpha_i \lambda^i$ for $\lambda \in \sigma(T)$, associate the operator

$$p(T) = \sum_{i=0}^{n} \alpha_i T^i.$$

Let $P'(\sigma(T))$ be the unital algebra of all polynomials (in one variable with complex coefficients) on $\sigma(T)$. The map $\Phi_T: P'(\sigma(T)) \to \mathcal{B}[\mathcal{H}]$ given by

$$\Phi_T(p) = p(T) \quad \text{for each} \quad p \in P'(\sigma(T))$$

is a unital homomorphism from $P'(\sigma(T))$ to $\mathcal{B}[\mathcal{H}]$. In fact, if $q(\lambda) = \sum_{j=0}^{m} \beta_j \lambda^j$, then $(pq)(\lambda) = \sum_{i=0}^{n} \sum_{j=0}^{m} \alpha_i \beta_j \lambda^{i+j}$ so that $\Phi_T(pq) = \sum_{i=0}^{n} \sum_{j=0}^{m} \alpha_i \beta_j T^{i+j} = \Phi_T(p)\Phi_T(q)$, and also $\Phi_T(1) = 1(T) = T^0 = I$. This can be extended from finite power series (i.e., from polynomials) to infinite power series: if a sequence $\{p_n\}$, with $p_n(\lambda) = \sum_{k=0}^{n} \alpha_k \lambda^k$ for each n, converges in some sense on $\sigma(T)$ to a limit ψ, denoted by $\psi(\lambda) = \sum_{k=0}^{\infty} \alpha_k \lambda^k$, then we may set $\psi(T) = \sum_{k=0}^{\infty} \alpha_k T^k$ if the series converges in some topology of $\mathcal{B}[\mathcal{H}]$. For instance, if $\alpha_k = \frac{1}{k!}$ for

each $k \geq 0$, then $\{p_n\}$ converges pointwise to the exponential function ψ such that $\psi(\lambda) = e^\lambda$ for every $\lambda \in \sigma(T)$ (on any $\sigma(T) \subseteq \mathbb{C}$), and so we define

$$e^T = \psi(T) = \sum_{k=0}^{\infty} \frac{1}{k!} T^k$$

(where the series converges uniformly in $\mathcal{B}[\mathcal{H}]$). However, unlike in the preceding paragraph, complex conjugation does not define an involution on $P'(\sigma(T))$ (and so $P'(\sigma(T))$ is not $*$-algebra under it). Indeed, $p^*(\lambda) = \overline{p(\lambda)}$ is not a polynomial in λ — the continuous function $\overline{p(\cdot)} : \sigma(T) \to \mathbb{C}$ given by $\overline{p(\lambda)} = \sum_{i=0}^{n} \overline{\alpha_i} \overline{\lambda}^i$ is not a polynomial in λ (for $\sigma(T) \not\subseteq \mathbb{R}$, even if $\sigma(T)^* = \sigma(T)$).

The extension of the mapping Φ_T to some larger algebras of functions is the focus of this chapter.

4.2 Functional Calculus for Normal Operators

Let $\mathcal{A}_{\sigma(T)}$ be the σ-algebra of Borel subsets of the spectrum $\sigma(T)$ of an operator T on a Hilbert space $\mathcal{H}$. Let $B(\sigma(T))$ be the Banach space of all bounded $\mathcal{A}_{\sigma(T)}$-measurable complex-valued functions $\psi : \sigma(T) \to \mathbb{C}$ on $\sigma(T)$ equipped with the sup-norm, and let $L^\infty(\sigma(T), \mu)$ be the Banach space of all essentially bounded (i.e., μ-essentially bounded) $\mathcal{A}_{\sigma(T)}$-measurable complex-valued functions $\psi : \sigma(T) \to \mathbb{C}$ on $\sigma(T)$, also equipped with the sup-norm $\| \cdot \|_\infty$, for any positive measure μ on $\mathcal{A}_{\sigma(T)}$. In both cases we have unital commutative C*-algebras, where involution is defined as complex conjugation, and the identity element is the characteristic function $1 = \chi_{\sigma(T)}$ of $\sigma(T)$ (carefully note that this is not the identity function $\varphi(\lambda) = \lambda \chi_{\sigma(T)}$ on $\sigma(T)$). If T is normal, then let $E : \mathcal{A}_{\sigma(T)} \to \mathcal{B}[\mathcal{H}]$ be the unique spectral measure of its spectral decomposition $T = \int \lambda \, dE_\lambda$ as in Theorem 3.15.

Theorem 4.2. *Let $T \in \mathcal{B}[\mathcal{H}]$ be a normal operator and consider its spectral decomposition $T = \int \lambda \, dE_\lambda$. For every function $\psi \in B(\sigma(T))$ there is a unique operator $\psi(T) \in \mathcal{B}[\mathcal{H}]$ given by*

$$\psi(T) = \int \psi(\lambda) \, dE_\lambda,$$

which is normal. Moreover, the mapping $\Phi_T : B(\sigma(T)) \to \mathcal{B}[\mathcal{H}]$ such that $\Phi_T(\psi) = \psi(T)$ is a unital $*$-homomorphism.

Proof. $\psi(T)$ is unique in $\mathcal{B}[\mathcal{H}]$ and normal for every $\psi \in \mathcal{B}(\sigma(T))$ by Lemmas 3.7 and 3.9. The mapping $\Phi_T : B(\sigma(T)) \to \mathcal{B}[\mathcal{H}]$ is a unital $*$-homomorphism by Lemma 3.8 and by the linearity of the integral. Indeed,

(a) $\Phi_T(\alpha \psi + \beta \phi) = (\alpha \psi + \beta \phi)(T) = \int (\alpha \psi(\lambda) + \beta \phi(\lambda)) \, dE_\lambda$
$\qquad\qquad = \alpha \psi(T) + \beta \phi(T) = \alpha \Phi_T(\psi) + \beta \Phi_T(\phi),$

(b) $\Phi_T(\psi \phi) = (\psi \phi)(T) = \int \psi(\lambda) \phi(\lambda) \, dE_\lambda = \psi(T) \phi(T) = \Phi_T(\psi) \Phi_T(\phi),$

(c) $\Phi_T(1) = 1(T) = \int dE_\lambda = I$,

(d) $\Phi_T(\psi^*) = \psi^*(T) = \int \overline{\psi(\lambda)} \, dE_\lambda = \psi(T)^* = \Phi_T(\psi)^*$,

for every $\psi, \phi \in \mathcal{B}(\sigma(T))$ and every $\alpha, \beta \in \mathbb{C}$, where the constant function 1 (i.e., $1(\lambda) = 1$ for every $\lambda \in \sigma(T)$) is the identity in $\mathcal{B}(\sigma(T))$.   □

Let $P(\sigma(T))$ be the set of all polynomials $p(\cdot,\cdot)\colon \sigma(T) \times \sigma(T) \to \mathbb{C}$ in λ and $\overline{\lambda}$ (or, $p(\cdot,\overline{\cdot})\colon \sigma(T) \to \mathbb{C}$ such that $\lambda \mapsto p(\lambda, \overline{\lambda})$), and let $C(\sigma(T))$ be the set of all complex-valued continuous (thus $\mathcal{A}_{\sigma(T)}$-measurable) functions on $\sigma(T)$. So

$$P'(\sigma(T)) \subset P(\sigma(T)) \subset C(\sigma(T)) \subset B(\sigma(T)),$$

where the last inclusion follows from the Weierstrass Theorem since $\sigma(T)$ is compact — cf. proof of Theorem 2.2. Except for $P'(\sigma(T))$, these are all unital $*$-subalgebras of $B(\sigma(T))$ and the restriction of the mapping Φ_T to any of them remains a unital $*$-homomorphism into $\mathcal{B}[\mathcal{H}]$. Thus Theorem 4.2 holds for all those unital $*$-subalgebras of the unital C*-algebra $B(\sigma(T))$. Note that $P'(\sigma(T))$ is a unital subalgebra of $P(\sigma(T))$, and the restriction of Φ_T to it is a unital homomorphism into $\mathcal{B}[\mathcal{H}]$, and so Theorem 4.2 also holds for $P'(\sigma(T))$ by replacing $*$-homomorphism with plain homomorphism. In particular, for p in $P'(\sigma(T))$, say $p(\lambda) = \sum_{i=0}^n \alpha_i \lambda^i$, we get, if T is normal,

$$p(T) = \int p(\lambda) \, dE_\lambda = \sum_{i=0}^n \alpha_i \int \lambda^i \, dE_\lambda = \sum_{i=0}^n \alpha_i T^i$$

and, for p in $P(\sigma(T))$, say $p(\lambda, \overline{\lambda}) = \sum_{i,j=0}^{n,m} \alpha_{i,j} \lambda^i \overline{\lambda}^j$ (cf. Lemma 3.10), we get

$$p(T, T^*) = \int p(\lambda, \overline{\lambda}) \, dE_\lambda = \sum_{i,j=0}^{n,m} \alpha_{i,j} \int \lambda^i \overline{\lambda}^j \, dE_\lambda = \sum_{i,j=0}^{n,m} \alpha_{i,j} T^i T^{*j}.$$

The quotient space of a set modulo an equivalence relation is a partition of the set. So we may regard $B(\sigma(T))$ as consisting of equivalence classes from $L^\infty(\sigma(T), \mu)$ — bounded functions are essentially bounded with respect to any measure μ. On the other hand, shifting from restriction to extension, it is sensible to argue that the preceding theorem might be extended from $B(\sigma(T))$ to some $L^\infty(\sigma(T), \mu)$, where bounded would be replaced by essentially bounded. But essentially bounded with respect to which measure μ?

A measure μ on a σ-algebra is *absolutely continuous* with respect to another measure ν on the same σ-algebra if $\nu(\Lambda) = 0$ implies $\mu(\Lambda) = 0$ for every measurable set Λ. Two measures on the same σ-algebra are *equivalent* if they are absolutely continuous with respect to each other, which means that they have the same sets of measure zero. That is, μ and ν are equivalent if, for every measurable set Λ, $\nu(\Lambda) = 0$ if and only if $\mu(\Lambda) = 0$. (Note: if two positive measures are equivalent, it may happen that one is finite but not the other.)

Remark. Let $E\colon \mathcal{A}_{\sigma(T)} \to \mathcal{B}[\mathcal{H}]$ be the spectral measure of the spectral decomposition of a normal operator T, and consider the statements of Lemmas 3.7, 3.8, and 3.9 for $\Omega = \sigma(T)$. *Lemma 3.7 remains true if $B(\sigma(T))$ is replaced by*

$L^\infty(\sigma(T), \mu)$ *for every positive measure* $\mu\colon \mathcal{A}_{\sigma(T)} \to \overline{\mathbb{R}}$ *that is equivalent to the spectral measure* E. The proof is essentially the same as that of Lemma 3.7. In fact, the spectral integrals of functions in $L^\infty(\sigma(T), \mu)$ coincide whenever they are equal μ-almost everywhere, and so a function in $B(\sigma(T))$ has the same integral of any representative from its equivalence class in $L^\infty(\sigma(T), \mu)$. Moreover, since μ and E have exactly the same sets of measure zero, we get

$$\|\phi\|_\infty \;=\; \operatorname{ess\,sup} |\phi| \;=\; \inf_{\Lambda \in \mathcal{N}_{\sigma(T)}} \; \sup_{\lambda \in \sigma(T) \setminus \Lambda} |\phi(\lambda)|$$

for every $\phi \in L^\infty(\sigma(T), \mu)$, where $\mathcal{N}_{\sigma(T)}$ denotes the collection of all $\Lambda \in \mathcal{A}_{\sigma(T)}$ such that $\mu(\Lambda) = 0$ or, equivalently, such that $E(\Lambda) = O$. Thus the sup-norm $\|\phi\|_\infty$ of $\phi\colon \sigma(T) \to \mathbb{C}$ with respect to μ dominates the sup-norm with respect to $\pi_{x,x}$ for every $x \in \mathcal{H}$. Then the proof of Lemma 3.7 remains unchanged if $B(\sigma(T))$ is replaced by $L^\infty(\sigma(T), \mu)$ for any positive measure μ equivalent to E, since $|f(x,x)| \le \|\phi\|_\infty \int d\pi_{x,x} = \|\phi\|_\infty \|x\|^2$ for every $x \in \mathcal{H}$ still holds if $\phi \in L^\infty(\sigma(T), \mu)$. Hence, for every $\phi \in L^\infty(\sigma(T), \mu)$, there is a unique operator $F \in \mathcal{B}[\mathcal{H}]$ such that $F = \int \phi(\lambda)\, dE_\lambda$ as in Lemma 3.7. This ensures that *Lemmas 3.8 and 3.9 also hold if* $B(\sigma(T))$ *is replaced by* $L^\infty(\sigma(T), \mu)$.

Theorem 4.3. *Let* $T \in \mathcal{B}[\mathcal{H}]$ *be a normal operator and consider its spectral decomposition* $T = \int \lambda\, dE_\lambda$. *If* μ *is a positive measure on* $\mathcal{A}_{\sigma(T)}$ *equivalent to the spectral measure* $E\colon \mathcal{A}_{\sigma(T)} \to \mathcal{B}[\mathcal{H}]$, *then for every* $\psi \in L^\infty(\sigma(T), \mu)$ *there is a unique operator* $\psi(T) \in \mathcal{B}[\mathcal{H}]$ *given by*

$$\psi(T) = \int \psi(\lambda)\, dE_\lambda,$$

which is normal. Moreover, the mapping $\Phi_T\colon L^\infty(\sigma(T), \mu) \to \mathcal{B}[\mathcal{H}]$ *such that* $\Phi_T(\psi) = \psi(T)$ *is a unital $*$-homomorphism.*

Proof. If $\mu\colon \mathcal{A}_{\sigma(T)} \to \overline{\mathbb{R}}$ and $E\colon \mathcal{A}_{\sigma(T)} \to \mathcal{B}[\mathcal{H}]$ are equivalent measures, then the above Remark ensures that the proof of Theorem 4.2 still holds. $\square$

There are however instances where it is convenient that a positive measure μ on $\mathcal{A}_{\sigma(T)}$, besides being equivalent to the spectral measure E, be finite as well. For example, if ψ lies in $L^\infty(\sigma(T), \mu)$ and μ is finite, then ψ also lies in $L^2(\sigma(T), \mu)$, since $\|\psi\|_2^2 = \int |\psi|^2 d\mu \le \|\psi\|_\infty^2\, \mu(\sigma(T)) \le \infty$; this is especially significant if it is expected that an essentially constant function will be in $L^2(\sigma(T), \mu)$, as will be the case in the proof of Lemma 4.7 below.

Definition 4.4. A *scalar spectral measure* for a normal operator is a positive *finite* measure equivalent to the spectral measure of its spectral decomposition.

In other words, if T is a normal operator and E is the unique spectral measure of Theorem 3.15, then a positive *finite* measure μ is a scalar spectral measure for T if, for every measurable set Λ, $\mu(\Lambda) = 0$ if and only if $E(\Lambda) = O$. Clearly (by transitivity), every positive finite measure that is equivalent to some scalar spectral measure for T is again a scalar spectral measure for T.

Definition 4.5. A *separating vector* for a collection $\{C_\gamma\}_{\gamma \in \Gamma} \subseteq \mathcal{B}[\mathcal{H}]$ of operators is a vector $e \in \mathcal{H}$ such that $e \notin \mathcal{N}(C)$ for every $O \neq C \in \{C_\gamma\}_{\gamma \in \Gamma}$.

Take the collection $\{E(\Lambda)\}_{\Lambda \in \mathcal{A}_\Omega}$ of all images of the spectral measure E associated to a normal operator T on $\mathcal{H}$, where Ω is either the disjoint union of $\{\Omega_\gamma\}$ or $\sigma(T)$ as in proof of Theorem 3.15. For each $x, y \in \mathcal{H}$, let $\pi_{x,y} \colon \mathcal{A}_\Omega \to \mathbb{C}$ be the scalar-valued (complex) measure of Section 3.3 related to E, namely,

$$\pi_{x,y}(\Lambda) = \langle E(\Lambda)x \,;\, y \rangle \quad \text{for every} \quad \Lambda \in \Omega.$$

Lemma 4.6. *If $T \in \mathcal{B}[\mathcal{H}]$ is normal, then $e \in \mathcal{H}$ is a separating vector for $\{E(\Lambda)\}_{\Lambda \in \mathcal{A}_\Omega}$ if and only if $\pi_{e,e}$ is a scalar spectral measure for T.*

Proof. Let $E \colon \mathcal{A}_\Omega \to \mathcal{B}[\mathcal{H}]$ be the spectral measure for a normal operator T in $\mathcal{B}[\mathcal{H}]$. Note that $\pi_{x,x}(\Lambda) = \|E(\Lambda)x\|^2$ for every $\Lambda \in \mathcal{A}_\Omega$ and every $x \in \mathcal{H}$, and so the measure $\pi_{x,x} \colon \mathcal{A}_\Omega \to \mathbb{R}$ is positive and finite for every $x \in \mathcal{H}$. Recall that $e \in \mathcal{H}$ is a separating vector for the collection $\{E(\Lambda)\}_{\Lambda \in \mathcal{A}_\Omega}$ if and only if $E(\Lambda)e \neq 0$ whenever $E(\Lambda) \neq O$, or equivalently, if and only if $\{E(\Lambda)e \neq 0 \iff E(\Lambda) \neq O\}$. Conclusion: $e \in \mathcal{H}$ is a separating vector for the collection $\{E(\Lambda)\}_{\Lambda \in \mathcal{A}_\Omega}$ if and only if $\{\pi_{e,e}(\Lambda) = 0 \iff E(\Lambda) = O\}$; that is, if and only if E and $\pi_{e,e}$ are equivalent measures. Since $\pi_{e,e}$ is positive and finite, this means that $\pi_{e,e}$ is a scalar spectral measure for T. $\qquad\square$

Lemma 4.7. *If T is a normal operator on a* separable *Hilbert space $\mathcal{H}$, then there is a separating vector $e \in \mathcal{H}$ for $\{E(\Lambda)\}_{\Lambda \in \mathcal{A}_\Omega}$, and so there is a vector $e \in \mathcal{H}$ such that $\pi_{e,e}$ is a scalar spectral measure for T (by Lemma 4.6).*

Proof. Take the positive measure μ on $\mathcal{A}_\Omega$ of Theorem 3.11. If $\mathcal{H}$ is separable, then μ is finite (Corollary 3.13). Let M_φ be the multiplication operator on $L^2(\Omega, \mu)$ with $\varphi \in L^\infty(\Omega, \mu)$ given by $\varphi(\lambda) = \lambda \in \sigma(T) \subseteq \mathbb{C}$ for every $\lambda \in \Omega$. Let $\chi_\Lambda \in L^\infty(\Omega, \mu)$ be the characteristic function of each $\Lambda \in \mathcal{A}_\Omega$. Set $E'(\Lambda) = M_{\chi_\Lambda}$ for every $\Lambda \in \mathcal{A}_\Omega$ so that $E' \colon \mathcal{A}_\Omega \to \mathcal{B}[L^2(\Omega, \mu)]$ is the spectral measure in $L^2(\Omega, \mu)$ for the spectral decomposition (cf. proof (a) of Theorem 3.15)

$$M_\varphi = \int \varphi(\lambda)\, dE'_\lambda = \int \lambda\, dE'_\lambda$$

of the normal operator $M_\varphi \in \mathcal{B}[L^2(\Omega, \mu)]$. Hence, for each $f \in L^2(\Omega, \mu)$,

$$\pi'_{f,f}(\Lambda) = \langle E'(\Lambda)f \,;\, f \rangle = \langle M_{\chi_\Lambda} f \,;\, f \rangle = \int \chi_\Lambda f \overline{f}\, d\mu = \int_\Lambda |f|^2 d\mu$$

for every $\Lambda \in \mathcal{A}_\Omega$. In particular, set $f = 1 = \chi_\Omega$ (the constant function $f(\lambda) = 1$ for all $\lambda \in \Omega$, i.e., the characteristic function χ_Ω of Ω), which lies in $L^2(\Omega, \mu)$ because μ is finite. Note that $f = 1 = \chi_\Omega$ is a separating vector for $\{E'(\Lambda)\}_{\Lambda \in \mathcal{A}_\Omega}$ since $E'(\Lambda)1 = M_{\chi_\Lambda} 1 = \chi_\Lambda \neq 0$ if $\Lambda \neq \varnothing$ in $\mathcal{A}_\Omega$. Thus $\pi'_{1,1}$ is a scalar spectral measure for M_φ by Lemma 4.6. Now, if $T \in \mathcal{B}[\mathcal{H}]$ is normal, then $T \cong M_\varphi$ by Theorem 3.11. Let $E \colon \mathcal{A}_\Omega \to \mathcal{B}[\mathcal{H}]$ be the spectral measure in $\mathcal{H}$ for T, given by $E(\Lambda) = U^* E'(\Lambda) U$ for each $\Lambda \in \mathcal{A}_\Omega$, where U is a unitary in $\mathcal{B}[\mathcal{H}, L^2(\Omega, \mu)]$

(cf. proof (b) of Theorem 3.15). Set $e = U^*1 \in \mathcal{H}$ with the function $1 = \chi_\Omega$ in $L^2(\Omega, \mu)$. Note that e is a separating vector for $\{E(\Lambda)\}_{\Lambda \in \mathcal{A}_\Omega}$ because $E(\Lambda)e = U^*E'(\Lambda)UU^*1 = U^*E'(\Lambda)1 \neq 0$ if $E(\Lambda) \neq O$ (since $E'(\Lambda)1 \neq 0$ when $E'(\Lambda) = UE(\Lambda)U^* \neq O$). Thus $\pi_{e,e}$ is a scalar spectral measure for T (Lemma 4.6). Next, take the spectral measure $\tilde{E} : \mathcal{A}_{\sigma(T)} \to \mathcal{B}[\mathcal{H}]$ such that $\tilde{E}(\Lambda) = E(\cup\Lambda)$ for every $\Lambda \in \mathcal{A}_{\sigma(T)}$ (cf. proof (c) of Theorem 3.15). Thus $\tilde{E}(\Lambda) \neq O$ implies that $\tilde{E}(\Lambda)e = E(\cup\Lambda)e \neq O$, and so e is also a separating vector for $\{\tilde{E}(\Lambda)\}_{\Lambda \in \mathcal{A}_{\sigma(T)}}$. Hence $\tilde{\pi}_{e,e}$, such that $\tilde{\pi}_{e,e}(\Lambda) = \langle \tilde{E}(\Lambda)e\,; e \rangle$, is a scalar spectral measure for T (Lemma 4.6). Again, use the same notation: E for $\tilde{E}$ and $\pi_{e,e}$ for $\tilde{\pi}_{e,e}$. $\square$

Remark. The preceding proof shows that the scalar spectral measure $\pi'_{1,1}$ for M_φ is precisely the measure μ of Theorem 3.11. Indeed, $\pi'_{1,1}(\Lambda) = \|E'(\Lambda)1\|^2 = \|M\chi_\Lambda 1\|^2 = \|\chi_\Lambda\|^2 = \int_\Lambda d\mu = \mu(\Lambda)$ for every $\Lambda \in \mathcal{A}_\Omega$. And so is the scalar spectral measure $\pi_{e,e}$ for T. That is, $\mu = \pi'_{1,1} = \pi_{e,e}$. In fact, for every $\Lambda \in \mathcal{A}_\Omega$,

$$\mu(\Lambda) = \pi'_{1,1}(\Lambda) = \langle E'(\Lambda)1\,; 1 \rangle = \langle UE(\Lambda)U^*Ue\,; Ue \rangle = \langle E(\Lambda)e\,; e \rangle = \pi_{e,e}(\Lambda).$$

Theorem 4.8. *Let $T \in \mathcal{B}[\mathcal{H}]$ be a normal operator on a separable Hilbert space $\mathcal{H}$ and consider its spectral decomposition $T = \int \lambda \, dE_\lambda$. Let μ be an arbitrary scalar spectral measure for T. For every $\psi \in L^\infty(\sigma(T), \mu)$ there is a unique operator $\psi(T) \in \mathcal{B}[\mathcal{H}]$ given by*

$$\psi(T) = \int \psi(\lambda) \, dE_\lambda,$$

which is normal. Moreover, the mapping $\Phi_T : L^\infty(\sigma(T), \mu) \to \mathcal{B}[\mathcal{H}]$ such that $\Phi_T(\psi) = \psi(T)$ is an isometric unital $*$-homomorphism.

Proof. All but the fact that Φ_T is an isometry is a particular case of Theorem 4.3. We show that Φ_T is an isometry. Lemma 3.8, the uniqueness in Lemma 3.7, and the uniqueness of the expression for $\psi(T)$ ensure that, for every $x \in \mathcal{H}$,

$$\|\Phi_T(\psi)x\|^2 = \langle \psi(T)^*\psi(T)x\,; x \rangle = \int \overline{\psi(\lambda)}\,\psi(\lambda)\, d\langle E_\lambda x\,; x \rangle = \int |\psi(\lambda)|^2 \, d\pi_{x,x}.$$

Suppose $\Phi_T(\psi) = O$. Then $\int |\psi(\lambda)|^2 \, d\pi_{x,x} = 0$ for every $x \in \mathcal{H}$. Lemma 4.7 ensures the existence of a separating vector $e \in \mathcal{H}$ for $\{E(\Lambda)\}_{\Lambda \in \mathcal{A}_{\sigma(T)}}$. Thus consider the scalar spectral measure $\pi_{e,e}$ so that $\int |\psi(\lambda)|^2 \, d\pi_{e,e} = 0$, and hence $\psi = 0$ in $L^\infty(\sigma(T), \pi_{e,e})$, which means that $\psi = 0$ in $L^\infty(\sigma(T), \mu)$ for every measure μ equivalent to $\pi_{e,e}$; in particular, for every scalar spectral measure μ for T. So $\{0\} = \mathcal{N}(\Phi_T) \subseteq L^\infty(\sigma(T), \mu)$, and hence the linear transformation $\Phi_T : L^\infty(\sigma(T), \mu) \to \mathcal{B}[\mathcal{H}]$ is injective; thus an isometry by Lemma 4.1(ℓ). $\square$

If T is normal on a *separable* Hilbert space $\mathcal{H}$, then $\Phi_T : L^\infty(\sigma(T), \mu) \to \mathcal{B}[\mathcal{H}]$ is an isometric unital $*$-homomorphism between C*-algebras (Theorem 4.8). Thus $\Phi_T : L^\infty(\sigma(T), \mu) \to \mathcal{R}(\Phi_T)$ is an isometric unital $*$-isomorphism between the C*-algebras $L^\infty(\sigma(T), \mu)$ and $\mathcal{R}(\Phi_T) \subseteq \mathcal{B}[\mathcal{H}]$ (see Remark after Lemma 4.1), which is the von Neumann algebra generated by T (see Proposition 4.A).

Theorem 4.9. *Let $T \in \mathcal{B}[\mathcal{H}]$ be a normal operator on a* separable *Hilbert space $\mathcal{H}$ and consider its spectral decomposition $T = \int \lambda \, dE_\lambda$. For every $\psi \in C(\sigma(T))$ there is a unique operator $\psi(T) \in \mathcal{B}[\mathcal{H}]$ given by*

$$\psi(T) = \int \psi(\lambda) \, dE_\lambda,$$

which is normal. Moreover, the mapping $\Phi_T \colon C(\sigma(T)) \to \mathcal{R}(\Phi_T) \subseteq \mathcal{B}[\mathcal{H}]$ such that $\Phi_T(\psi) = \psi(T)$ is an isometric unital $*$-isomorphism, where $\mathcal{R}(\Phi_T)$ is the C*-algebra generated by T and the identity I in $\mathcal{B}[\mathcal{H}]$.

Proof. The set $C(\sigma(T))$ is a unital $*$-subalgebra of the C*-algebra $B(\sigma(T))$, which can be viewed as a subalgebra of the C*-algebra $L^\infty(\sigma(T), \mu)$ for every measure μ (in terms of equivalence classes in $L^\infty(\sigma(T), \mu)$ of each function in $B(\sigma(T))$). So $\Phi_T \colon C(\sigma(T)) \to \mathcal{B}[\mathcal{H}]$, the restriction of $\Phi_T \colon L^\infty(\sigma(T), \mu) \to \mathcal{B}[\mathcal{H}]$ to $C(\sigma(T))$, is an isometric unital $*$-homomorphism under the same assumptions of Theorem 4.8. Hence $\Phi_T \colon C(\sigma(T)) \to \mathcal{R}(\Phi_T)$ is an isometric unital $*$-isomorphism (cf. the paragraph that precedes the theorem statement). Thus it remains to show that $\mathcal{R}(\Phi_T)$ is the C*-algebra generated by T and the identity I. Indeed, take $\mathcal{R}(\Phi_T) = \Phi_T(C(\sigma(T)))$. Since $\Phi_T(P(\sigma(T))) = \mathcal{P}(T, T^*)$, the algebra of all polynomials in T and T^*, since $P(\sigma(T))^- = C(\sigma(T))$ in the sup-norm topology by the Stone–Weierstrass Theorem (cf. proof of Theorem 3.11), and $\Phi_T \colon C(\sigma(T)) \to \mathcal{B}[\mathcal{H}]$ is an isometry (thus with a closed range), it follows that $\Phi_T(C(\sigma(T))) = \Phi_T(P(\sigma(T))^-) = \Phi_T(P(\sigma(T)))^- = \mathcal{P}(T, T^*)^- = \mathcal{C}^*(T)$, where $\mathcal{C}^*(T)$ stands for the C*-algebra generated by T and I (see the paragraph following Definition 3.12, recalling that now T is normal). $\qquad\square$

The mapping Φ_T of Theorems 4.2, 4.3, 4.8, and 4.9 such that $\Phi_T(\psi) = \psi(T)$ is referred to as the *functional calculus* for T, and the theorems themselves are referred to as the FUNCTIONAL CALCULUS FOR NORMAL OPERATORS. Since $\sigma(T)$ is compact it follows that, for any positive measure μ on $\mathcal{A}_{\sigma(T)}$,

$$C(\sigma(T)) \subset B(\sigma(T)) \subset L^\infty(\sigma(T), \mu)$$

(again, the last inclusion is interpreted in terms of the equivalence classes in $L^\infty(\sigma(T), \mu)$ of each representative in $B(\sigma(T))$, as in the Remark that precedes Theorem 4.3). Thus ψ lies in $L^\infty(\sigma(T), \mu)$ as in Theorem 4.3 or 4.8 whenever ψ satisfies the assumptions of Theorems 4.2 and 4.9, and so the next result applies to any of the preceding forms of the Functional Calculus Theorems. It says that $\psi(T)$ *commutes with every operator that commutes with T*.

Corollary 4.10. *Let $T = \int \lambda \, dE_\lambda$ be a normal operator in $\mathcal{B}[\mathcal{H}]$. Take any ψ in $L^\infty(\sigma(T), \mu)$, where μ is any positive measure on $\mathcal{A}_{\sigma(T)}$ equivalent to the spectral measure E. Consider the operator $\psi(T)$ of Theorem 4.3, viz.,*

$$\psi(T) = \int \psi(\lambda) \, dE_\lambda$$

in $\mathcal{B}[\mathcal{H}]$. If S in $\mathcal{B}[\mathcal{H}]$ commutes with T, then S commutes with $\psi(T)$.

Proof. If $ST = TS$, then by Theorem 3.17 (cf. proof of Corollary 3.19),

$$\langle S\,\psi(T)x\,;y\rangle = \langle \psi(T)x\,;S^*y\rangle = \int \psi(\lambda)\,d\langle E_\lambda x\,;S^*y\rangle$$

$$= \int \psi(\lambda)\,d\langle SE_\lambda x\,;y\rangle = \int \psi(\lambda)\,d\langle E_\lambda Sx\,;y\rangle = \langle \psi(T)Sx\,;y\rangle$$

for every $x, y \in \mathcal{H}$ (see Lemma 3.7), which means that $S\psi(T) = \psi(T)S$. $\quad\square$

If T is a normal operator, then the Spectral Mapping Theorem for polynomials in Theorem 2.8 does not depend on the Spectral Theorem. Observe that, for a normal operator, the Spectral Mapping Theorem for polynomials in Theorem 2.7 is a particular case of Theorem 2.8. However, extensions from polynomials to bounded or continuous functions do depend on the Spectral Theorem via Theorems 4.8 or 4.9 as we shall see next. The forthcoming Theorems 4.11 and 4.12 are also referred to as SPECTRAL MAPPING THEOREMS for bounded or continuous functions of normal operators.

Theorem 4.11. *If T is a normal operator on a* separable *Hilbert space, if μ is a scalar spectral measure for T, and if ψ lies in $L^\infty(\sigma(T), \mu)$, then*

$$\sigma(\psi(T)) = \operatorname{ess}\mathcal{R}(\psi) = \operatorname{ess}\big\{\psi(\lambda) \in \mathbb{C}\colon \lambda \in \sigma(T)\big\} = \operatorname{ess}\psi(\sigma(T)).$$

Proof. If the theorem holds for a scalar spectral measure, then it holds for any scalar spectral measure. Suppose T is a normal operator on a separable Hilbert space. Let μ be the positive measure of Theorem 3.11, which is a scalar spectral measure for T (cf. Remark that follows the proof of Lemma 4.7). Now set $\tilde{\mu} = \tilde{\pi}_{e,e}$ as in the proof of Theorem 4.7, which is again a scalar spectral measure for T and, as before, use the same notation for μ and $\tilde{\mu}$). Take any ψ in $L^\infty(\sigma(T), \mu)$. By Theorem 4.8, $\Phi_T\colon L^\infty(\sigma(T), \mu) \to \mathcal{B}[\mathcal{H}]$ is an isometric (thus injective) unital $*$-isomorphism between C*-algebras and $\Phi_T(\psi) = \psi(T)$. So

$$\sigma(\psi) = \sigma(\Phi_T(\psi)) = \sigma(\psi(T))$$

by Lemma 4.1(h). Consider the multiplication operator $\mathcal{M}_\varphi$ on the Hilbert space $L^2(\Omega, \mu)$ of Theorem 3.11. Since $T \cong M_\varphi$, $L^2(\Omega, \mu)$ is separable. Since $\varphi(\Omega)^- = \sigma(T)$ (with $\varphi(\lambda) = \lambda$), take $\psi \circ \varphi \in L^\infty(\Omega, \mu)$ (also denoted by ψ) so that the spectrum $\sigma(\psi)$ coincides in both algebras. Since μ is a scalar spectral measure for $\mathcal{M}_\varphi$ (cf. Remark that follows the proof of Lemma 4.7 again), we may apply Theorem 4.8 to M_φ. Therefore, using the same argument,

$$\sigma(\psi) = \sigma(\psi(M_\varphi)).$$

Claim. $\quad \psi(M_\varphi) = M_\psi.$

Proof. Let E' be the spectral measure for M_φ, where $E'(\Lambda) = M_{\chi_\Lambda}$ with χ_Λ being the characteristic function of Λ in $\mathcal{A}_\Omega$ as in the proof of Lemma 4.7. Take the scalar-valued measure $\pi'_{f,g}$ given by $\pi'_{f,g}(\Lambda) = \langle E'(\Lambda)f\,;g\rangle$ for every Λ in $\mathcal{A}_\Omega$ and each $f, g \in L^2(\Omega, \mu)$. Recall from the remark that follows the proof of Lemma 4.7 that $\mu = \pi'_{1,1}$. Observe that, for every Λ in $\mathcal{A}_\Omega$,

$$\int_\Lambda d\,\pi'_{f,g} = \pi'_{f,g}(\Lambda) = \langle E'(\Lambda)f\,;g\rangle = \langle \chi_\Lambda f\,;g\rangle = \int_\Lambda f\,\overline{g}\,d\mu = \int_\Lambda f\,\overline{g}\,d\,\pi'_{1,1}.$$

So $d\,\pi'_{f,g} = f\overline{g}\,d\,\pi'_{1,1}$ ($f\overline{g}$ is the Radon–Nikodým derivative of $\pi'_{f,g}$ with respect to $\pi'_{1,1}$; see proof of Lemma 3.8). Since $\psi(M_\varphi) = \int \psi(\lambda)\,dE'_\lambda$ by Theorem 4.8,

$$\langle \psi(M_\varphi)f\,;g\rangle = \int \psi\,d\,\langle E'_\lambda f\,;g\rangle = \int \psi\,d\,\pi'_{f,g}$$

$$= \int \psi f\overline{g}\,d\,\pi'_{1,1} = \int \psi f\overline{g}\,d\mu = \langle M_\psi f\,;g\rangle$$

for every $f,g \in L^2(\Omega,\mu)$, and so $\psi(M_\phi) = M_\psi$, proving the claimed identity.

Hence, since μ is finite (thus σ-finite), it follows by Proposition 3.B that

$$\sigma(\psi(T)) = \sigma(\psi(M_\varphi)) = \sigma(M_\psi) = \operatorname{ess}\mathcal{R}(\psi). \qquad \square$$

Theorem 4.12. *If T is a normal operator on a* separable *Hilbert space, then*

$$\sigma(\psi(T)) = \psi(\sigma(T)) \quad \text{for every} \quad \psi \in C(\sigma(T)).$$

Proof. Recall that $C(\sigma(T)) \subset B(\sigma(T)) \subset L^\infty(\sigma(T),\mu)$ for every measure μ. Take an arbitrary $\psi \in C(\sigma(T))$. Since $\sigma(T)$ is compact in $\mathbb{C}$, and ψ is continuous, it follows that $\mathcal{R}(\psi) = \psi(\sigma(T))$ is compact (see, e.g., [66, Theorem 4.64]), and so $\mathcal{R}(\psi)^- = \mathcal{R}(\psi)$. Consider the setup of the previous proof: $\tilde{\mu} = \tilde{\pi}_{e,e}$ is also denoted by μ (the positive measure of Theorem 3.11), which is a scalar spectral measure for T and for $M_\varphi \cong T$. From Proposition 3.B we get

$$\operatorname{ess}\mathcal{R}(\psi) = \big\{\alpha \in \mathbb{C}\colon \mu(\psi^{-1}(B_\varepsilon(\alpha))) > 0 \text{ for all } \varepsilon > 0\big\} \subseteq \mathcal{R}(\psi),$$

where $B_\varepsilon(\alpha)$ stands for the open ball of radius ε centered at α, and also that $\mu(\Lambda) > 0$ for every nonempty $\Lambda \in \mathcal{A}_{\sigma(T)}$ open relative to $\sigma(T) = \sigma(M_\varphi)$. All that has been said so far about μ holds for every positive measure equivalent to it; in particular, it holds for every scalar spectral measure for T. If there is an α in $\mathcal{R}(\psi)\backslash\operatorname{ess}\mathcal{R}(\psi)$, then there is an $\varepsilon > 0$ such that $\mu(\psi^{-1}(B_\varepsilon(\alpha))) = 0$. Since ψ is continuous, the inverse image $\psi^{-1}(B_\varepsilon(\alpha))$ of the open set $B_\varepsilon(\alpha)$ in $\mathbb{C}$ must be open in $\sigma(T)$, and hence $\psi^{-1}(B_\varepsilon(\alpha)) = \varnothing$. But this is a contradiction because, since $\alpha \in \mathcal{R}(\psi)$, it follows that $\varnothing \neq \psi^{-1}(\{\alpha\}) \subseteq \psi^{-1}(B_\varepsilon(\alpha))$. Thus $\mathcal{R}(\psi)\backslash\operatorname{ess}\mathcal{R}(\psi) = \varnothing$, and so $\mathcal{R}(\psi) = \operatorname{ess}\mathcal{R}(\psi)$. Therefore, by Theorem 4.11,

$$\sigma(\psi(T)) = \operatorname{ess}\mathcal{R}(\psi) = \mathcal{R}(\psi) = \psi(\sigma(T)). \qquad \square$$

4.3 Analytic Functional Calculus: Riesz Functional Calculus

The approach of this section is rather different from that of Section 4.2. However, Proposition 4.J will show that the Riesz functional calculus, when restricted to a normal operator on a Hilbert space, coincides with the functional calculus for normal operators of the previous section (for analytic functions).

Measure theory was the main ingredient for the last four sections of Chapter 3 and also for Section 2 of this chapter. The background for the present section is complex analysis, where the integrals involved here are all Riemann's (unlike the measure theoretic integrals we have been dealing with so far). For the standard results of complex analysis used in this section the reader is referred, for instance, to [1], [13], [21], [26], and [79].

An *arc* is a *continuous* function from an arbitrary (nondegenerate) closed interval of the real line into the complex plane, say, $\alpha:[0,1] \to \mathbb{C}$, where $\Gamma = \mathcal{R}(\alpha) = \{\alpha(t) \in \mathbb{C}: t \in [0,1]\}$ is the range of the arc α, which is connected and compact (continuous image of connected sets is connected, and continuous image of compact sets is compact). Let $\mathbb{G} = \{\#X: \varnothing \neq X \in \wp([0,1])\}$ be the set of all cardinal numbers of nonempty subsets of the interval $[0,1]$. Given an arc $\alpha:[0,1] \to \Gamma$ set $\mu(t) = \#\{s \in [0,1): \alpha(s) = \alpha(t)\} \in \mathbb{G}$ for each $t \in [0,1)$. That is, $\mu(t) \in \mathbb{G}$ is the cardinality of the subset of $[0,1]$ consisting of all points $s \in [0,1)$ such that $\alpha(s) = \alpha(t)$, for every $t \in [0,1)$. We say that $\mu(t)$ is the *multiplicity of the point* $\alpha(t) \in \Gamma$. Consider the function $\mu:[0,1) \to \mathbb{G}$ defined by $t \mapsto \mu(t)$, which is referred to as the *multiplicity of the arc* α. To avoid space-filling curves, assume from now on that Γ is nowhere dense. A pair (Γ, μ) consisting of the range Γ and the multiplicity μ of an arc is called a *directed pair* if the direction in which the arc traverses its range Γ, according to the natural order of the interval $[0,1]$, is taken into account. An *oriented curve* generated by an arc $\alpha:[0,1] \to \Gamma$ is the directed pair (Γ, μ), and the arc α itself is referred to as a *parameterization* of the curve (Γ, μ) (which is not unique for (Γ, μ)). If an oriented curve (Γ, μ) is such that $\alpha(0) = \alpha(1)$ for some parameterization α (and so for all parameterizations), then it is called a *closed curve*. A parameterization α of an oriented curve (Γ, μ) is injective on $[0,1)$ if and only if $\mu = 1$ (i.e., $\mu(t) = 1$ for all $t \in [0,1)$). If one parameterization of $(\Gamma, 1)$ is injective, then so are all of them. A curve $(\Gamma, 1)$ parameterized by an arc α that is injective on $[0,1)$ is called a *simple curve* (or a *Jordan curve*).

Thus an oriented curve consists of a directed pair (Γ, μ), where $\Gamma \subset \mathbb{C}$ is the (nowhere dense) range of a continuous function $\alpha:[0,1] \to \mathbb{C}$ called a *parameterization* of (Γ, μ), and μ is a $\mathbb{G}$-valued function that assigns to each point t of $[0,1)$ the multiplicity of the point $\alpha(t)$ in Γ (i.e., each $\mu(t)$ says how often α traverses the point $\gamma = \alpha(t)$ of Γ). For notational simplicity we shall refer to Γ itself as an oriented curve when the multiplicity μ is either clear in the context or is immaterial. In this case we simple say "Γ is an oriented curve".

By a *partition P of the interval* $[0,1]$ we mean a finite sequence $\{t_j\}_{j=0}^{n}$ of points in $[0,1]$ such that $0 = t_0$, $t_{j-1} < t_j$ for $1 \leq j \leq n$, and $t_n = 1$. The *total variation* of a function $\alpha:[0,1] \to \mathbb{C}$ is the supremum of the set $\{\nu \in \mathbb{R}: \nu = \sum_{j=1}^{n} |\alpha(t_j) - \alpha(t_{j-1})|\}$ taken over all partitions P of $[0,1]$. A function $\alpha:[0,1] \to \mathbb{C}$ is of *bounded variation* if its total variation is finite. An oriented curve Γ is *rectifiable* if some (and so any) parameterization α of it is of bounded variation, and its *length* $\ell(\Gamma)$ is the total variation of α. An arc $\alpha:[0,1] \to \mathbb{C}$ is

continuously differentiable (or *smooth*) if it is differentiable on the open interval $(0, 1)$, has one-sided derivatives at the end points 0 and 1, and its derivative $\alpha':[0, 1] \to \mathbb{C}$ is continuous on $[0, 1]$. In this case the oriented curve Γ parameterized by α is also referred to as *continuously differentiable* (or *smooth*). The linear space of all continuously differentiable functions from $[0, 1]$ to $\mathbb{C}$ is usually denoted by $C^1([0, 1])$. If there is a partition of $[0, 1]$ such that each restriction $\alpha|_{[t_{j-1}, t_j]}$ of the continuous α is continuously differentiable (i.e., such that $\alpha|_{[t_{j-1}, t_j]}$ lies in $C^1([t_{j-1}, t_j])$), then $\alpha:[0, 1] \to \mathbb{C}$ is a *piecewise continuously differentiable arc* (or *piecewise smooth arc*), and Γ parameterized by it is an oriented *piecewise continuously differentiable curve* (or an oriented *piecewise smooth curve*). In this case it is known from advanced calculus that Γ is rectifiable (α is of bounded variation) with length $\ell(\Gamma) = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} |\alpha'_{[t_{j-1}, t_j]}(t)|\, dt$.

A continuous function $\psi:\Gamma \to \mathbb{C}$ has a Riemann–Stieltjes integral associated with parameterization $\alpha:[0, 1] \to \Gamma$ of a rectifiable curve Γ,

$$\int_{\Gamma} \psi(\lambda)\, d\lambda = \int_{0}^{1} \psi(\alpha(t))\, d\alpha(t).$$

If, in addition, α is smooth, then

$$\int_{0}^{1} \psi(\alpha(t))\, d\alpha(t) = \int_{0}^{1} \psi(\alpha(t))\, \alpha'(t)\, dt.$$

Now consider a *bounded* function $f:\Gamma \to \mathcal{Y}$, where $\mathcal{Y}$ is a Banach space. Let $\alpha:[0, 1] \to \Gamma$ be a parameterization of an oriented curve Γ, and let $P = \{t_j\}_{j=0}^{n}$ be a partition of the interval $[0, 1]$. Recall that the norm of P is the number $\|P\| = \max_{1 \le j \le n}(t_j - t_{j-1})$. A Riemann–Stieltjes sum for the function f with respect to α based on a given partition P is a vector

$$S(f, \alpha, P) = \sum_{j=1}^{n} f(\alpha(\tau_j))\, (\alpha(t_j) - \alpha(t_{j-1})) \ \in \ \mathcal{Y}$$

with $\tau_j \in [t_{j-1}, t_j]$. Suppose there is a unique vector $S \in \mathcal{Y}$ with the following property: for every $\varepsilon > 0$ there is a $\delta_\varepsilon > 0$ such that, if P is an arbitrary partition of $[0, 1]$ with $\|P\| \le \delta_\varepsilon$, then $\|S(f, \alpha, P) - S\| \le \varepsilon$ for all Riemann–Stieltjes sums $S(f, \alpha, P)$ for f with respect to α based on P. When such a vector S exists, it is called the *Riemann–Stieltjes integral* of f with respect to α, denoted by

$$S = \int_{\Gamma} f(\lambda)\, d\lambda = \int_{0}^{1} f(\alpha(t))\, d\alpha(t).$$

If the $\mathcal{Y}$-valued function $f:\Gamma \to \mathcal{Y}$ is *continuous*, and if Γ is *rectifiable*, then it can be shown (exactly as in the case of complex-valued functions; see, e.g., [21, Proposition 5.3 and p. 385]) that there exists the *Riemann–Stieltjes integral* of f with respect to α. If α_1 and α_2 are parameterizations of Γ, then for every $\delta > 0$ there are partitions P_1 and P_2 with norm less than δ such that $S(f, \alpha_1, P_1) = S(f, \alpha_2, P_2)$. Given a rectifiable oriented curve Γ and a continuous $\mathcal{Y}$-valued function on Γ, then the integral $\int_{\Gamma} f(\lambda)\, d\lambda$ does not depend on the parameterization of Γ, and we shall refer to it as *the integral of f over Γ*.

Let $C[\Gamma, \mathcal{Y}]$ and $B[\Gamma, \mathcal{Y}]$ be the linear spaces of all $\mathcal{Y}$-valued continuous or bounded functions on Γ, respectively, equipped with the sup-norm, which are Banach spaces because Γ is compact and $\mathcal{Y}$ is complete. Recall that, since Γ is compact, then $C[\Gamma, \mathcal{Y}] \subseteq B[\Gamma, \mathcal{Y}]$. It is readily verified that the transformation $\int \colon C[\Gamma, \mathcal{Y}] \to \mathcal{Y}$, assigning to each f in $C[\Gamma, \mathcal{Y}]$ its integral in $\mathcal{Y}$, is linear and bounded (i.e., linear and continuous — $\int \in \mathcal{B}[C[\Gamma, \mathcal{Y}], \mathcal{Y}]$). Indeed,

$$\left\| \int_\Gamma f(\lambda)\, d\lambda \right\| \le \sup_{\lambda \in \Gamma} \|f(\lambda)\| \int_0^1 |d\alpha(t)| = \|f\|_\infty\, \ell(\Gamma),$$

and therefore, for every $L \in \mathcal{B}[\mathcal{Y}, \mathcal{Z}]$, where $\mathcal{Z}$ also is a Banach space,

$$L \int_\Gamma f(\lambda)\, d\lambda = \int_\Gamma L f(\lambda)\, d\lambda.$$

The *reverse* (or the *opposite*) of an oriented curve Γ is an oriented curve denoted by $-\Gamma$ obtained from Γ by reversing the orientation of Γ. To reverse the orientation of Γ means to reverse the order of the domain $[0, 1]$ of its parameterization α or, equivalently, to replace $\alpha(t)$ with $\alpha(-t)$ for t running over $[-1, 0]$. Reversing the orientation of a curve Γ has the effect of replacing $\int_\Gamma f(\lambda)\, d\lambda = \int_0^1 f(\alpha(t))\, d\alpha(t)$ with $\int_1^0 f(\alpha(t))\, d\alpha(t) = \int_{-\Gamma} f(\lambda)\, d\lambda$ so that

$$\int_{-\Gamma} f(\lambda)\, d\lambda = -\int_\Gamma f(\lambda)\, d\lambda.$$

Also note that the preceding arguments and results all remain valid if the continuous function f is defined on a subset Λ of $\mathbb{C}$ that includes the curve Γ (by simply replacing it with its restriction to Γ, which remains continuous). In particular, if Λ is a nonempty open subset of $\mathbb{C}$ and Γ is a rectifiable curve included in Λ, and if $f \colon \Lambda \to \mathcal{Y}$ is a continuous function on the open set Λ, then f has an integral over $\Gamma \subset \Lambda \subseteq \mathbb{C}$, viz., $\int_\Gamma f(\lambda)\, d\lambda$.

Recall that a topological space is *disconnected* if it is the union of two disjoint nonempty subsets that are both open and closed. Otherwise, the topological space is said to be *connected*. A subset of a topological space is called a *connected set* (or a *connected subset*) if, as a topological subspace, it is a connected topological space itself. Take any nonempty subset Λ of $\mathbb{C}$. A *component* of Λ is a maximal connected subset of Λ, which coincides with the union of all connected subsets of Λ containing a given point of Λ. Thus any two components of Λ are disjoint. Since the closure of a connected set is connected, any component of Λ is closed relative to Λ. By a *region* (or a *domain*) we mean a nonempty connected open subset of $\mathbb{C}$. Every open subset of $\mathbb{C}$ is uniquely expressed as a countable union of disjoint regions that are components of it (see, e.g., [21, Proposition 3.9]). The closure of a region is sometimes called a *closed region*. Carefully note that different regions may have the same closure (sample: a punctured open disk).

If Γ is a closed rectifiable oriented curve in $\mathbb{C}$ and if $\zeta \in \mathbb{C} \backslash \Gamma$, then a classical and important result in complex analysis says that the integral

$$w_\Gamma(\zeta) = \tfrac{1}{2\pi i} \int_\Gamma \frac{1}{\lambda - \zeta} \, d\lambda$$

has an integer value that is constant on each component of $\mathbb{C}\backslash\Gamma$ and zero on the unbounded component of $\mathbb{C}\backslash\Gamma$. The number $w_\Gamma(\zeta)$ is referred to as the *winding number* of Γ about ζ. *If a closed rectifiable oriented curve Γ is a simple curve, then $\mathbb{C}\backslash\Gamma$ has only two components, just one of then is bounded, and Γ is their common boundary* (this is the Jordan Curve Theorem). In this case (i.e., for a *simple* closed rectifiable oriented curve Γ), $w_\Gamma(\zeta)$ takes on only two values for each $\zeta \in \mathbb{C}\backslash\Gamma$, either 0 or ± 1, and if for every ζ in the bounded component of $\mathbb{C}\backslash\Gamma$ we get $w_\Gamma(\zeta) = 1$, then we say that Γ is *positively* (i.e., counterclockwise) *oriented* and, otherwise, if $w_\Gamma(\zeta) = -1$, then we say that Γ is *negatively* (i.e., clockwise) *oriented*; and for every ζ in the unbounded component of $\mathbb{C}\backslash\Gamma$ we get $w_\Gamma(\zeta) = 0$. If Γ is positively oriented, then the reverse curve $-\Gamma$ is negatively oriented, and vice versa. These notions can be extended as follows. A finite union $\Gamma = \bigcup_{j=1}^{m} \Gamma_j$ of *disjoint* closed rectifiable oriented *simple* curves Γ_j is called a *path*, and its *winding number* $w_\Gamma(\zeta)$ about $\zeta \in \mathbb{C}\backslash\Gamma$ is defined by $w_\Gamma(\zeta) = \sum_{j=1}^{m} w_{\Gamma_j}(\zeta)$. A path Γ is *positively oriented* if for every $\zeta \in \mathbb{C}\backslash\Gamma$ the winding number $w_\Gamma(\zeta)$ is either 0 or 1, and *negatively oriented* if for every $\zeta \in \mathbb{C}\backslash\Gamma$ the winding number $w_\Gamma(\zeta)$ is either 0 or -1. If a path $\Gamma = \bigcup_{j=1}^{m} \Gamma_j$ is positively oriented, then the reverse path $-\Gamma = \bigcup_{j=1}^{m} -\Gamma_j$ is negatively oriented. The *inside* (notation: ins Γ) and the *outside* (notation: out Γ) of a positively oriented path Γ are the sets

$$\text{ins } \Gamma = \big\{ \zeta \in \mathbb{C} \colon w_\Gamma(\zeta) = 1 \big\} \quad \text{and} \quad \text{out } \Gamma = \big\{ \zeta \in \mathbb{C} \colon w_\Gamma(\zeta) = 0 \big\}.$$

From now on all paths are positively oriented. Note that if $\Gamma = \bigcup_{j=1}^{m} \Gamma_j$ is a path and if there is a finite subset $\{\Gamma_{j'}\}_{j'=1}^{n}$ of Γ such that $\Gamma_{j'} \subset \text{ins } \Gamma_{j'+1}$, then these nested (disjoint closed rectifiable oriented) simple curves $\{\Gamma_{j'}\}$ are oppositely oriented, with Γ_n being positively oriented, because $w_\Gamma(\zeta)$ is either 0 or 1 for every $\zeta \in \mathbb{C}\backslash\Gamma$. An open subset of $\mathbb{C}$ is a *Cauchy domain* (or a *Jordan domain*) if it has finitely many components whose closures are pairwise disjoint, and its boundary is a path. The closure of a Jordan domain is sometimes referred to as a *Jordan closed region*. Also observe that if Γ is a path, then $\{\Gamma, \text{ins } \Gamma, \text{out } \Gamma\}$ is a partition of $\mathbb{C}$. Since a path Γ is a closed set in $\mathbb{C}$ (finite union of closed sets), and since it is the common boundary of ins Γ and out Γ, it follows that *ins Γ and out Γ are open sets in $\mathbb{C}$*, and their closures are given by the union with their common boundary,

$$(\text{ins } \Gamma)^- = \Gamma \cup \text{ins } \Gamma \quad \text{and} \quad (\text{out } \Gamma)^- = \Gamma \cup \text{out } \Gamma.$$

If Γ is a path, if $\mathcal{Y}$ is a Banach space, and if $f \colon \Gamma \to \mathcal{Y}$ is a continuous function (so that f has an integral over each Γ_j), then we define the integral of the $\mathcal{Y}$-valued f over the path $\Gamma \subset \mathbb{C}$ by

$$\int_\Gamma f(\lambda) \, d\lambda = \sum_{j=1}^{m} \int_{\Gamma_j} f(\lambda) \, d\lambda.$$

Again, if $f\colon \Lambda \to \mathcal{Y}$ is a continuous function on a nonempty open subset Λ of $\mathbb{C}$ that includes a path Γ, then we define the integral of f over the path Γ as the integral of the restriction of f to Γ over $\Gamma \subset \Lambda \subseteq \mathbb{C}$; that is, as the integral of $f|_\Gamma \colon \Gamma \to \mathcal{Y}$ (which is continuous as well) over the path Γ:

$$\int_\Gamma f(\lambda)\, d\lambda = \int_\Gamma f|_\Gamma(\lambda)\, d\lambda.$$

Therefore, if we are given a nonempty open subset Λ of $\mathbb{C}$, a Banach space $\mathcal{Y}$, and a continuous function $f\colon \Lambda \to \mathcal{Y}$, then we have seen how to define the integral of f over any path Γ included in Λ.

The CAUCHY INTEGRAL FORMULA says that *if $\psi\colon \Lambda \to \mathbb{C}$ is an analytic function on a nonempty open subset Λ of $\mathbb{C}$ that includes a path Γ and its inside ins Γ (i.e., $\Gamma \cup \text{ins}\,\Gamma \subset \Lambda \subseteq \mathbb{C}$) then, for every $\zeta \in \text{ins}\,\Gamma$,*

$$\psi(\zeta) = \tfrac{1}{2\pi i} \int_\Gamma \frac{\psi(\lambda)}{\lambda - \zeta}\, d\lambda.$$

In fact, this is a most common application of the general case of the Cauchy Integral Formula (see, e.g., [21, Problem 5.O]). Observe that the assumption $\Gamma \cup \text{ins}\,\Gamma \subset \Lambda \subseteq \mathbb{C}$ (i.e., $(\text{ins}\,\Gamma)^- \subset \Lambda \subseteq \mathbb{C}$) simply means that $\Gamma \subset \Lambda \subseteq \mathbb{C}$ and $\mathbb{C}\backslash\Lambda \subset \text{out}\,\Gamma$ (i.e., $w_\Gamma(\zeta) = 0$ for every $\zeta \in \mathbb{C}\backslash\Lambda$), and the assumption $\zeta \in \text{ins}\,\Gamma \subset \Lambda$ is equivalent to saying that $\zeta \in \Lambda\backslash\Gamma$ and $w_\Gamma(\zeta) = 1$. Since the function $\psi(\cdot)[(\cdot) - \zeta]^{-1}\colon \Gamma \to \mathbb{C}$ is continuous on the path Γ, it follows that the above integral may be generalized to a $\mathcal{Y}$-valued function.

Thus consider the following special case in a complex Banach space $\mathcal{Y}$. Throughout this chapter $\mathcal{X}$ will denote a nonzero complex Banach space. Set $\mathcal{Y} = \mathcal{B}[\mathcal{X}]$, the Banach algebra of all operators on $\mathcal{X}$. Take any operator T in $\mathcal{B}[\mathcal{X}]$ and let $R_T\colon \rho(T) \to \mathcal{G}[\mathcal{X}]$ be its resolvent function, which is defined by $R_T(\lambda) = (\lambda I - T)^{-1}$ for every λ in the resolvent set $\rho(T)$, which in turn is an open and nonempty subset of $\mathbb{C}$ (cf. Section 2.1). Let $\psi\colon \Lambda \to \mathbb{C}$ be an analytic function on a nonempty open subset Λ of $\mathbb{C}$ such that $\Lambda \cap \rho(T)$ is nonempty. Recall from the proof of Theorem 2.2 that $R_T\colon \rho(T) \to \mathcal{G}[\mathcal{X}]$ is continuous, and so is the product $\psi R_T\colon \Lambda \cap \rho(T) \to \mathcal{B}[\mathcal{X}]$, which is defined by $(\psi R_T)(\lambda) = \psi(\lambda) R_T(\lambda) \in \mathcal{B}[\mathcal{X}]$ for every λ in the open subset $\Lambda \cap \rho(T)$ of $\mathbb{C}$. Then we can define the integral of ψR_T over any path $\Gamma \subset \Lambda \cap \rho(T)$,

$$\int_\Gamma \psi(\lambda)\, R_T(\lambda)\, d\lambda = \int_\Gamma \psi(\lambda)\, (\lambda I - T)^{-1}\, d\lambda \ \in \ \mathcal{B}[\mathcal{X}].$$

That is, the integral of the $\mathbb{C}$-valued function $\psi(\cdot)[(\cdot) - \zeta]^{-1}\colon \Lambda\backslash\{\zeta\} \to \mathbb{C}$, for each $\zeta \in \text{ins}\,\Gamma$, on any path Γ such that $(\text{ins}\,\Gamma)^- \subset \Lambda$, is generalized to the $\mathcal{B}[\mathcal{X}]$-valued function $\psi(\cdot)[(\cdot)I - T]^{-1}\colon \Lambda \cap \rho(T) \to \mathcal{B}[\mathcal{X}]$, for each $T \in \mathcal{B}[\mathcal{X}]$, on any path Γ such that $\Gamma \subset \Lambda \cap \rho(T)$.

We shall see below that ψR_T in fact is analytic, where the definition of a Banach-space-valued analytic function on a nonempty open subset of $\mathbb{C}$ is exactly the same as that for a scalar-valued analytic function.

If Ω is a compact set included in an open subset Λ of $\mathbb{C}$ (so $\Omega \subset \Lambda \subseteq \mathbb{C}$), then there exists a path $\Gamma \subset \Lambda$ such that $\Omega \subset \text{ins } \Gamma$ and $\mathbb{C}\backslash\Lambda \subset \text{out } \Gamma$ (see, e.g., [27, p. 200]). As we have seen before, this is equivalent to saying that $\Omega \subset \text{ins } \Gamma \subset (\text{ins } \Gamma)^- \subset \Lambda$. Take an operator $T \in \mathcal{B}[\mathcal{X}]$. Set $\Omega = \sigma(T)$, which is compact. Now take an open subset Λ of $\mathbb{C}$ such that $\sigma(T) \subset \Lambda$. Let Γ be any path such that $\Gamma \subset \Lambda$ and $\sigma(T) \subset \text{ins } \Gamma \subset \Lambda$. That is, for a given open set Λ, let the path Γ satisfy

$$\sigma(T) \subset \text{ins } \Gamma \subset (\text{ins } \Gamma)^- \subset \Lambda.$$

Since $\rho(T) = \mathbb{C}\backslash\sigma(T)$, the above inclusions ensure that $\Gamma \subset \Lambda \cap \rho(T) \neq \varnothing$.

Lemma 4.13. *If $\psi\colon \Lambda \to \mathbb{C}$ is an analytic function on a nonempty open subset Λ of $\mathbb{C}$, then the integral*

$$\int_{\Gamma} \psi(\lambda)\, R_T(\lambda)\, d\lambda = \int_{\Gamma} \psi(\lambda)\, (\lambda I - T)^{-1}\, d\lambda$$

does not depend on the choice of any path Γ that satisfies the assumption

$$\sigma(T) \subset \text{ins } \Gamma \subset (\text{ins } \Gamma)^- \subset \Lambda.$$

Proof. Let T be an arbitrary operator in $\mathcal{B}[\mathcal{X}]$ and take its resolvent function $R_T\colon \rho(T) \to \mathcal{G}[\mathcal{X}]$ so that $R_T(\lambda) = (\lambda I - T)^{-1}$ for every $\lambda \in \rho(T)$. Let $\psi\colon \Lambda \to \mathbb{C}$ be an analytic function on an open set $\Lambda \subseteq \mathbb{C}$ that properly includes $\sigma(T)$, so that there is a path Γ satisfying the above assumption. If there was only one of such a path, then there was nothing to prove. Thus suppose Γ' and Γ'' are distinct paths satisfying the above assumption. We shall show that

$$\int_{\Gamma'} \psi(\lambda)\, R_T(\lambda)\, d\lambda = \int_{\Gamma''} \psi(\lambda)\, R_T(\lambda)\, d\lambda.$$

Claim 1. The product $\psi R_T\colon \Lambda \cap \rho(T) \to \mathcal{B}[\mathcal{X}]$ is analytic on $\Lambda \cap \rho(T)$.

Proof. First note that $\Lambda \cap \rho(T) \neq \varnothing$. The resolvent identity of Section 2.1 ensures that if λ and ν are distinct points in $\rho(T)$, then

$$\frac{R_T(\lambda) - R_T(\nu)}{\lambda - \nu} + R_T(\nu)^2 = \big(R_T(\nu) - R_T(\lambda)\big) R_T(\nu)$$

(cf. proof of Theorem 2.2). Since $R_T\colon \rho(T) \to \mathcal{G}[\mathcal{X}]$ is continuous,

$$\lim_{\lambda \to \nu} \left\| \frac{R_T(\lambda) - R_T(\nu)}{\lambda - \nu} + R_T(\nu)^2 \right\| \leq \lim_{\lambda \to \nu} \left\| R_T(\nu) - R_T(\lambda) \right\| \left\| R_T(\nu) \right\| = 0$$

for every $\nu \in \rho(T)$. Thus the resolvent function $R_T\colon \rho(T) \to \mathcal{G}[\mathcal{X}]$ is analytic on $\rho(T)$ and so it is analytic on the nonempty open set $\Lambda \cap \rho(T)$. Since the function $\psi\colon \Lambda \to \mathbb{C}$ also is analytic on $\Lambda \cap \rho(T)$, and since the product of analytic functions on the same domain is again analytic, it follows that the product

function $\psi R_T \colon \Lambda \cap \rho(T) \to \mathcal{B}[\mathcal{X}]$ such that $(\psi R_T)(\lambda) = \psi(\lambda) R_T(\lambda) \in \mathcal{B}[\mathcal{X}]$ for every $\lambda \in \Lambda \cap \rho(T)$ is analytic on $\Lambda \cap \rho(T)$, concluding the proof of Claim 1.

The CAUCHY THEOREM says that *if $\psi \colon \Lambda \to \mathbb{C}$ is analytic on a nonempty open subset Λ of $\mathbb{C}$ that includes a path Γ and its inside* ins Γ, *then*

$$\int_\Gamma \psi(\lambda)\, d\lambda = 0$$

(see, e.g., [21, Problem 5.O]). This can be extended from scalar-valued functions to Banach-space-valued functions. Let $\mathcal{Y}$ be a complex Banach space, let $f \colon \Lambda \to \mathcal{Y}$ be an analytic function on a nonempty open subset Λ of $\mathbb{C}$ that includes a path Γ and its inside ins Γ, and consider the integral $\int_\Gamma f(\lambda)$.

Claim 2. If $f \colon \Lambda \to \mathcal{Y}$ is analytic on a given nonempty open subset Λ of $\mathbb{C}$ and if Γ is any path such that $(\mathrm{ins}\ \Gamma)^- \subset \Lambda$, then

$$\int_\Gamma f(\lambda)\, d\lambda = 0.$$

Proof. Take an arbitrary nonzero η in $\mathcal{Y}^*$ (i.e., take a nonzero bounded linear functional $\eta \colon \mathcal{Y} \to \mathbb{C}$), and consider the composition $\eta \circ f \colon \Lambda \to \mathbb{C}$ of η with an analytic function $f \colon \Lambda \to \mathcal{Y}$, which is again analytic on Λ. Indeed,

$$\left| \frac{\eta(f(\lambda)) - \eta(f(\nu))}{\lambda - \nu} - \eta(f'(\nu)) \right| \le \|\eta\| \left| \frac{f(\lambda) - f(\nu)}{\lambda - \nu} - f'(\nu) \right|$$

for every pair of distinct points λ and ν in Λ. Since both η and the integral are linear and continuous,

$$\eta\!\left(\int_\Gamma f(\lambda)\, d\lambda \right) = \eta\!\left(\sum_{j=1}^m \int_{\Gamma_j} f(\lambda)\, d\lambda \right) = \sum_{j=1}^m \eta\!\left(\int_{\Gamma_j} f(\lambda)\, d\lambda \right)$$

$$= \sum_{j=1}^m \int_{\Gamma_j} \eta(f(\lambda))\, d\lambda = \int_\Gamma \eta(f(\lambda))\, \lambda.$$

The Cauchy Theorem for scalar-valued functions says that $\int_\Gamma \eta(f(\lambda))\, d\lambda = 0$ (because $\eta \circ f$ is analytic on Λ). Therefore, $\eta(\int_\Gamma f(\lambda)\, d\lambda) = 0$. Since this holds for every $\eta \in \mathcal{Y}^*$, the Hahn–Banach Theorem (see, e.g., [66, Corollary 4.64]) ensures the claimed result, viz., $\int_\Gamma f(\lambda) = 0$.

Take the nonempty open subset ins $\Gamma' \cap$ ins Γ'' of $\mathbb{C}$, which includes $\sigma(T)$. Let Δ be the open component of ins $\Gamma' \cap$ ins Γ'' including $\sigma(T)$. Consider the (rectifiable positively oriented) simple curve $\Gamma \subseteq \Gamma' \cup \Gamma''$ consisting of the boundary of Δ, and so ins $\Gamma = \Delta$ and $\Delta^- = (\mathrm{ins}\ \Gamma)^- = \Delta \cup \Gamma$. Observe that

$$\sigma(T) \subset \mathrm{ins}\ \Gamma \subset (\mathrm{ins}\ \Gamma)^- \subseteq (\mathrm{ins}\ \Gamma' \cap \mathrm{ins}\ \Gamma'')^- \subset \Lambda.$$

Thus $\Gamma \subset \Delta \cap \rho(T) \ne \varnothing$. Set $\widetilde{\Delta} = \mathrm{ins}\ \Gamma' \backslash \Delta$ (nonempty, since Γ' and Γ'' are distinct). Orient the boundary of $\widetilde{\Delta}$ so as to make it a (positively oriented) path

$\widetilde{\Gamma} \subseteq \Gamma' \cup (-\Gamma)$ so that ins $\widetilde{\Gamma} = \widetilde{\Delta}$. Since ins $\Gamma' = \Delta \cup \widetilde{\Delta}$ with $\Delta \cap \widetilde{\Delta} = \varnothing$ and Γ', Γ, and $\widetilde{\Gamma}$ are positively oriented, $\int_{\Gamma'} f(\lambda)\,d\lambda = \int_{\Gamma} f(\lambda)\,d\lambda + \int_{\widetilde{\Gamma}} f(\lambda)\,d\lambda$ for every $\mathcal{B}[\mathcal{X}]$-valued continuous function f whose domain includes $(\text{ins }\Gamma')^-$. So

$$\int_{\Gamma'} \psi(\lambda)\,R_T(\lambda)\,d\lambda = \int_{\Gamma} \psi(\lambda)\,R_T(\lambda)\,d\lambda + \int_{\widetilde{\Gamma}} \psi(\lambda)\,R_T(\lambda)\,d\lambda.$$

But Claim 1 ensures that ψR_T is analytic on $\Lambda \cup \rho(T)$ and therefore, since $\widetilde{\Delta}^- = (\text{ins }\widetilde{\Gamma})^- \subset \Lambda \cup \rho(T)$, Claim 2 ensures that $\int_{\widetilde{\Gamma}} \psi(\lambda)\,R_T(\lambda)\,d\lambda = 0$. Hence

$$\int_{\Gamma'} \psi(\lambda)\,R_T(\lambda)\,d\lambda = \int_{\Gamma} \psi(\lambda)\,R_T(\lambda)\,d\lambda.$$

Similarly (by exactly the same argument),

$$\int_{\Gamma''} \psi(\lambda)\,R_T(\lambda)\,d\lambda = \int_{\Gamma} \psi(\lambda)\,R_T(\lambda)\,d\lambda. \qquad \square$$

Definition 4.14. Take an arbitrary $T \in \mathcal{B}[\mathcal{X}]$. If $\psi \colon \Lambda \to \mathbb{C}$ is analytic on an open set $\Lambda \subseteq \mathbb{C}$ that includes the spectrum $\sigma(T)$ of T and if Γ is any path such that $\sigma(T) \subset \text{ins }\Gamma \subset (\text{ins }\Gamma)^- \subset \Lambda$, then define the operator $\psi(T)$ in $\mathcal{B}[\mathcal{X}]$ by

$$\psi(T) = \tfrac{1}{2\pi i} \int_{\Gamma} \psi(\lambda)\,R_T(\lambda)\,d\lambda = \tfrac{1}{2\pi i} \int_{\Gamma} \psi(\lambda)\,(\lambda I - T)^{-1}\,d\lambda.$$

After defining the integral of ψR_T over any path Γ such that $\Gamma \subset \Lambda \cap \rho(T)$, and after ensuring in Lemma 4.13 its invariance for any path Γ such that $\sigma(T) \subset \text{ins }\Gamma \subset (\text{ins }\Gamma)^- \subset \Lambda$, then it is clear that the preceding definition of the operator $\psi(T)$ is motivated by the Cauchy Integral Formula.

Given an operator T in $\mathcal{B}[\mathcal{X}]$, we say that a complex-valued function ψ is *analytic on $\sigma(T)$* (or *analytic on a neighborhood of $\sigma(T)$*) if it is analytic on an open set that includes $\sigma(T)$. By a *path about $\sigma(T)$* we mean a path Γ whose inside (properly) includes $\sigma(T)$ (i.e., a path such that $\sigma(T) \subset \text{ins }\Gamma$). Thus what is behind Definition 4.14 is that, if ψ is analytic on $\sigma(T)$, then the above integral exists as an operator in $\mathcal{B}[\mathcal{X}]$, and the integral does not depend on the path about $\sigma(T)$ whose closure of its inside is included in the open set upon which ψ is defined (i.e., it is included in the domain of ψ).

Lemma 4.15. *If ϕ and ψ are analytic on $\sigma(T)$ and if Γ is any path about $\sigma(T)$ such that $\sigma(T) \subset \text{ins }\Gamma \subset (\text{ins }\Gamma)^- \subset \Lambda$, where the nonempty open set $\Lambda \subseteq \mathbb{C}$ is the intersection of the domains of ϕ and ψ, then*

$$\phi(T)\,\psi(T) = \tfrac{1}{2\pi i} \int_{\Gamma} \phi(\lambda)\,\psi(\lambda)\,R_T(\lambda)\,d\lambda,$$

so that $\phi(T)\,\psi(T) = (\phi\,\psi)(T)$ since $\phi\,\psi$ in analytic on $\sigma(T)$.

Proof. Let ϕ and ψ be analytic on $\sigma(T)$. Thus $\phi\,\psi$ is analytic on the intersection Λ of their domains. Let Γ and Γ' be arbitrary paths such that

$$\sigma(T) \subset \text{ins } \Gamma \subset (\text{ins } \Gamma)^- \subset \text{ins } \Gamma' \subseteq \Lambda.$$

Thus, by Definition 4.14 and by the resolvent identity (cf. Section 2.1),

$$\phi(T)\psi(T) = \left(\tfrac{1}{2\pi i}\int_\Gamma \phi(\lambda)\,R_T(\lambda)\,d\lambda\right)\left(\tfrac{1}{2\pi i}\int_{\Gamma'}\psi(\nu)\,R_T(\nu)\,d\nu\right)$$

$$= -\tfrac{1}{4\pi^2}\int_\Gamma\int_{\Gamma'}\phi(\lambda)\,\psi(\nu)\,R_T(\lambda)\,R_T(\nu)\,d\nu\,d\lambda$$

$$= -\tfrac{1}{4\pi^2}\int_\Gamma\int_{\Gamma'}\phi(\lambda)\,\psi(\nu)\,(\lambda-\nu)^{-1}\big(R_T(\lambda)-R_T(\nu)\big)\,d\nu\,d\lambda$$

$$= -\tfrac{1}{4\pi^2}\int_\Gamma\int_{\Gamma'}\phi(\lambda)\,\psi(\nu)\,(\lambda-\nu)^{-1}R_T(\lambda)\,d\nu\,d\lambda$$

$$\quad + \tfrac{1}{4\pi^2}\int_{\Gamma'}\int_\Gamma\phi(\lambda)\,\psi(\nu)\,(\lambda-\nu)^{-1}R_T(\nu)\,d\lambda\,d\nu$$

$$= -\tfrac{1}{4\pi^2}\int_\Gamma\phi(\lambda)\Big(\int_{\Gamma'}\psi(\nu)(\lambda-\nu)^{-1}\,d\nu\Big)R_T(\lambda)\,d\lambda$$

$$\quad + \tfrac{1}{4\pi^2}\int_{\Gamma'}\psi(\nu)\Big(\int_\Gamma\phi(\lambda)(\lambda-\nu)^{-1}\,d\lambda\Big)R_T(\nu)\,d\nu.$$

Since $\lambda \in \Gamma$, it follows that $\lambda \in \text{ins } \Gamma'$, and hence $\int_{\Gamma'}\psi(\nu)(\lambda-\nu)^{-1}\,d\nu = (2\pi i)\psi(\lambda)$ by the Cauchy Integral Formula. Moreover, since $\nu \in \Gamma'$, it follows that $\nu \in \text{out } \Gamma$ (so $\phi(\lambda)(\lambda-\nu)^{-1}$ is analytic on $\text{ins } \Gamma'$ which includes Γ), and hence $\int_\Gamma \phi(\lambda)(\lambda-\nu)^{-1}\,d\lambda = 0$ by the Cauchy Theorem. Therefore,

$$\phi(T)\psi(T) = \tfrac{1}{2\pi i}\int_\Gamma \phi(\lambda)\,\psi(\lambda)\,R_T(\lambda)\,d\lambda. \qquad \square$$

Given an operator $T \in \mathcal{B}[\mathcal{X}]$, let $A(\sigma(T))$ denote the set of all *analytic functions on the spectrum* $\sigma(T)$ of T. That is, $\psi \in A(\sigma(T))$ if $\psi\colon \Lambda \to \mathbb{C}$ is analytic on an open set $\Lambda \subseteq \mathbb{C}$ that includes $\sigma(T)$. It is readily verified that $A(\sigma(T))$ is a unital algebra, where the domain of the product of two functions in $A(\sigma(T))$ is the intersection of their domains, and the identity element 1 in $A(\sigma(T))$ is the constant function $1(\lambda) = 1$ for all $\lambda \in \Lambda$. Note that the identity function also lies in $A(\sigma(T))$ (i.e., if $\varphi\colon \Lambda \to \Lambda \subseteq \mathbb{C}$ is such that $\varphi(\lambda) = \lambda$ for every $\lambda \in \Lambda$, then $\varphi \in A(\sigma(T))$). The next theorem is the main result of this section. (Recall that $\mathcal{X}$ is an arbitrary nonzero complex Banach space.)

Theorem 4.16. Riesz Functional Calculus. *Take any operator $T \in \mathcal{B}[\mathcal{X}]$. For every function $\psi \in A(\sigma(T))$, let the operator $\psi(T) \in \mathcal{B}[\mathcal{X}]$ be defined as in Definition 4.14, viz.,*

$$\psi(T) = \tfrac{1}{2\pi i}\int_\Gamma \psi(\lambda)\,R_T(\lambda)\,d\lambda.$$

The mapping $\Phi_T\colon A(\sigma(T)) \to \mathcal{B}[\mathcal{X}]$ such that $\Phi_T(\psi) = \psi(T)$ is a homomorphism. Moreover,

$$\Phi_T(p) = p(T) = \tfrac{1}{2\pi i} \int_\Gamma \sum_{j=0}^{m} \alpha_j \, \lambda^j \, R_T(\lambda) \, d\lambda = \sum_{j=0}^{m} \alpha_j \, T^j$$

for every polynomial $p(\lambda) = \sum_{j=0}^{m} \alpha_j \lambda^j$ in one variable with complex coefficients (i.e., for every $p \in P'(\sigma(T)) \subset A(\sigma(T))$). In particular,

$$\Phi_T(1) = 1(T) = \tfrac{1}{2\pi i} \int_\Gamma R_T(\lambda) \, d\lambda = T^0 = I,$$

where 1 denotes the constant function, $1(\lambda) = 1$ for all λ in a neighborhood of $\sigma(T)$, so that Φ_T takes the identity element 1 of $A(\sigma(T))$ to the identity operator I in $\mathcal{B}[\mathcal{X}]$, and hence Φ_T is a unital homomorphism; and

$$\Phi_T(\varphi) = \varphi(T) = \tfrac{1}{2\pi i} \int_\Gamma \lambda \, R_T(\lambda) \, d\lambda = T,$$

where φ is the identity function: $\varphi(\lambda) = \lambda$ for λ in a neighborhood of $\sigma(T)$. Furthermore, if ψ has a power series expansion $\psi(\lambda) = \sum_{k=0}^{\infty} \alpha_k \lambda^k$ with radius of convergence greater than $r(T)$, and so on a neighborhood of $\sigma(T)$, then

$$\Phi_T(\psi) = \psi(T) = \tfrac{1}{2\pi i} \int_\Gamma \sum_{k=0}^{\infty} \alpha_k \, \lambda^k \, R_T(\lambda) \, d\lambda = \sum_{k=0}^{\infty} \alpha_k \, T^k,$$

where the series $\sum_{k=0}^{\infty} \alpha_k T^k$ converges in $\mathcal{B}[\mathcal{X}]$. Actually, if $\{\psi_n\}$ is a sequence in $A(\sigma(T))$ that converges uniformly on every compact subset of its domain Λ (such that $\sigma(T) \subset \mathrm{ins}\,\Gamma \subset (\mathrm{ins}\,\Gamma)^- \subset \Lambda$) to $\psi \in A(\sigma(T))$, then $\{\Phi_T(\psi_n)\}$ converges in $\mathcal{B}[\mathcal{X}]$ to $\Phi_T(\psi)$. Thus $\Phi_T \colon (A(\sigma(T)), \|\cdot\|_\infty) \to \mathcal{B}[\mathcal{X}]$ is continuous.

Proof. Since the integral $\int_\Gamma (\,\cdot\,) R_T \, d\lambda \colon A(\sigma(T)) \to \mathcal{B}[\mathcal{H}]$ is a linear transformation of the linear space $A(\sigma(T))$ into the linear space $\mathcal{B}[\mathcal{X}]$, it follows that $\Phi_T \colon A(\sigma(T)) \to \mathcal{B}[\mathcal{X}]$ is a linear transformation. That is,

$$\Phi_T(\alpha\,\phi + \beta\,\psi) = (\alpha\,\phi + \beta\,\psi)(T) = \tfrac{1}{2\pi i} \int_\Gamma (\alpha\,\phi + \beta\,\psi)(\lambda) \, R_T(\lambda) \, d\lambda$$

$$= \alpha \, \tfrac{1}{2\pi i} \int_\Gamma \phi(\lambda) \, R_T(\lambda) \, d\lambda + \beta \, \tfrac{1}{2\pi i} \int_\Gamma \psi(\lambda) \, R_T(\lambda) \, d\lambda$$

$$= \alpha\,\phi(T) + \beta\,\psi(T) = \alpha\,\Phi_T(\phi) + \beta\,\Phi_T(\psi)$$

for every $\alpha, \beta \in \mathbb{C}$ and every $\phi, \psi \in A(\sigma(T))$. Moreover, by Lemma 4.15,

$$\Phi_T(\phi\,\psi) = (\phi\,\psi)(T) = \tfrac{1}{2\pi i} \int_\Gamma (\phi\,\psi)(\lambda) \, R_T(\lambda) \, d\lambda$$

$$= \tfrac{1}{2\pi i} \int_\Gamma \phi(\lambda)\,\psi(\lambda) \, R_T(\lambda) \, d\lambda = \phi(T)\,\psi(T) = \Phi_T(\phi)\,\Phi_T(\psi)$$

for every $\phi, \psi \in A(\sigma(T))$. Thus Φ is a homomorphism. Recall: the preceding integrals do not depend on the path Γ such that $\sigma(T) \subset \mathrm{ins}\,\Gamma \subset (\mathrm{ins}\,\Gamma)^- \subset \Lambda$. Suppose ψ is analytic on the whole complex plane (i.e., suppose $\Lambda = \mathbb{C}$, which is the case if ψ is a polynomial), and let Γ be any circle about the origin with radius δ greater than the spectral radius of T; that is, $r(T) < \delta$. Since $r(T) < |\lambda|$ for every $\lambda \in \Gamma$, it follows by Corollary 2.12(a) that

$$(2\pi i)\,\psi(T) = \int_\Gamma \psi(\lambda)\,R_T(\lambda)\,d\lambda = \int_\Gamma \psi(\lambda)\sum_{k=0}^\infty \tfrac{1}{\lambda^{k+1}}\,T^k\,d\lambda$$

$$= I\int_\Gamma \psi(\lambda)\,\tfrac{1}{\lambda}\,d\lambda + \sum_{k=1}^\infty T^k\int_\Gamma \psi(\lambda)\,\tfrac{1}{\lambda^{k+1}}\,d\lambda$$

for every $\psi \in A(\sigma(T))$, where the above series converges uniformly (i.e., it converges in the operator norm topology of $\mathcal{B}[\mathcal{X}]$). Next consider two special cases consisting of elementary polynomials. First let $\psi = 1$ so that

$$1(T) = I\tfrac{1}{2\pi i}\int_\Gamma \tfrac{1}{\lambda}\,d\lambda + \sum_{k=1}^\infty T^k\,\tfrac{1}{2\pi i}\int_\Gamma \tfrac{1}{\lambda^{k+1}}\,d\lambda \;=\; I+0 \;=\; I,$$

since $\tfrac{1}{2\pi i}\int_\Gamma \tfrac{1}{\lambda}\,d\lambda = 1(0) = 1$ by the Cauchy Integral Formula and, for $k \geq 1$, $\int_\Gamma \tfrac{1}{\lambda^{k+1}}\,d\lambda = 0$. Indeed, with $\alpha(\theta) = \rho e^{i\theta}$ for every $\theta \in [0, 2\pi]$ we get $\int_\Gamma \tfrac{1}{\lambda}\,d\lambda = \int_0^{2\pi}\alpha(\theta)^{-1}\alpha'(\theta)\,d\theta = \int_0^{2\pi}\rho^{-1}e^{-i\theta}\rho\,i\,e^{i\theta}\,d\theta = i\int_0^{2\pi}d\theta = 2\pi i$ and, for $k \geq 1$, we also get $\int_\Gamma \tfrac{1}{\lambda^{k+1}}\,d\lambda = \int_0^{2\pi}\alpha(\theta)^{-(k+1)}\alpha'(\theta)\,d\theta = \int_0^{2\pi}\rho^{-(k+1)}e^{-i(k+1)\theta}\,i\rho\,e^{i\theta}\,d\theta = i\rho^{-k}\int_0^{2\pi}e^{-ik\theta}\,d\theta = 0$. Now let $\psi = \varphi$, the identity function, so that

$$\varphi(T) = I\tfrac{1}{2\pi i}\int_\Gamma d\lambda + T\tfrac{1}{2\pi i}\int_\Gamma \tfrac{1}{\lambda}\,d\lambda + \sum_{k=2}^\infty T^k\,\tfrac{1}{2\pi i}\int_\Gamma \tfrac{1}{\lambda^k}\,d\lambda = 0+T+0 = T,$$

since $\tfrac{1}{2\pi i}\int_\Gamma d\lambda = 0$ by the Cauchy Theorem (i.e., $\tfrac{1}{2\pi i}\int_\Gamma d\lambda = \int_0^{2\pi}\alpha'(\theta)\,d\theta = i\rho\int_0^{2\pi}e^{i\theta}\,d\theta = 0$) and the other two integrals were computed above. Thus

$$T^2 = \varphi(T)^2 = \tfrac{1}{2\pi i}\int_\Gamma \varphi(\lambda)^2 R_T(\lambda)\,d\lambda = \tfrac{1}{2\pi i}\int_\Gamma \lambda^2 R_T(\lambda)\,d\lambda$$

by Lemma 4.15, and so a trivial induction ensures that

$$T^j = \tfrac{1}{2\pi i}\int_\Gamma \lambda^j R_T(\lambda)\,d\lambda$$

for every integer $j \geq 0$. Therefore, by linearity of the integral,

$$p(T) = \sum_{j=0}^m \alpha_j T^j = \tfrac{1}{2\pi i}\int_\Gamma p(\lambda)\,R_T(\lambda)\,d\lambda$$

whenever $p(\lambda) = \sum_{j=0}^m \alpha_j\lambda^j$; that is, for every polynomial $p \in P'(\sigma(T))$. Next we extend this result from finite power series (i.e., from polynomials) to infinite power series. To say that a function ψ has a power series expansion $\psi(\lambda) = \sum_{k=0}^\infty \alpha_k\lambda^k$ on a neighborhood of $\sigma(T)$ means that the radius of convergence of the series $\sum_{k=0}^\infty \alpha_k\lambda^k$ is greater than $r(T)$, which implies that the polynomials $p_n(\lambda) = \sum_{k=0}^n \alpha_k\lambda^k$ make a sequence $\{p_n\}_{n=0}^\infty$ that converges uniformly to ψ on a closed disk whose boundary is a circle Γ of radius $r(T)+\varepsilon$ about the origin for some $\varepsilon > 0$, which in turn implies that $\psi \in A(\sigma(T))$. Thus, since the integral is linear and continuous (i.e., since $\int \in \mathcal{B}[C[\Gamma, \mathcal{B}[\mathcal{X}]], \mathcal{B}[\mathcal{X}]]$), and according to Corollary 2.12(a), it follows that

$$\psi(T) = \sum_{k=0}^\infty \alpha_k T^k.$$

Indeed,

$$\psi(T) = \tfrac{1}{2\pi i} \int_\Gamma \psi(\lambda)\, R_T(\lambda)\, d\lambda = \tfrac{1}{2\pi i} \int_\Gamma \left(\sum_{k=0}^{\infty} \alpha_k \lambda^k \right) R_T(\lambda)\, d\lambda$$

$$= \sum_{k=0}^{\infty} \alpha_k \left(\tfrac{1}{2\pi i} \int_\Gamma \lambda^k\, R_T(\lambda)\, d\lambda \right) = \sum_{k=0}^{\infty} \alpha_k\, T^k,$$

where the operator series $\sum_{k=0}^{\infty} \alpha_k T^k$ converges uniformly (i.e., it converges in the operator norm topology to $\psi(T) \in \mathcal{B}[\mathcal{X}]$). In fact, since the scalar series $\sum_{k=0}^{\infty} \alpha_k \lambda^k$ converges uniformly on Γ to $\psi \in A(\sigma(T))$,

$$\|\psi(T) - p_n(T)\| = \tfrac{1}{2\pi i} \left\| \int_\Gamma \big(\psi(\lambda) - p_n(\lambda) \big) R_T(\lambda)\, d\lambda \right\|$$

$$\leq \sup_{\lambda \in \Gamma} \big| \psi(\lambda) - p_n(\lambda) \big| \, \left\| \tfrac{1}{2\pi i} \int_\Gamma R_T(\lambda)\, d\lambda \right\|$$

$$= \sup_{\lambda \in \Gamma} \big| \psi(\lambda) - p_n(\lambda) \big| \to 0 \quad \text{as} \quad n \to \infty.$$

Hence $\Phi_T(p_n) \to \Phi_T(\psi)$ in $\mathcal{B}[\mathcal{X}]$. The same argument shows that if $\{\psi_n\}_{n=0}^{\infty}$ is a sequence of functions in $A(\sigma(T))$ that converges uniformly on every compact subset of its domain to $\psi \in A(\sigma(T))$, then $\Phi_T(\psi_n) \to \Phi_T(\psi)$ in $\mathcal{B}[\mathcal{X}]$. $\square$

Corollary 4.17. *Take any operator in $T \in \mathcal{B}[\mathcal{X}]$. If $\psi \in A(\sigma(T))$, then $\psi(T)$ commutes with every operator that commutes with T.*

Proof. This is the counterpart of Corollary 4.10. Let S be an operator in $\mathcal{B}[\mathcal{X}]$.

Claim. If $ST = TS$, then $R_T(\lambda)S = SR_T(\lambda)$ for every $\lambda \in \rho(T)$.

Proof. $S = S(\lambda I - T)(\lambda I - T)^{-1} = (\lambda I - T)S(\lambda I - T)^{-1}$ if $ST = TS$, and so $R_T(\lambda)S = (\lambda I - T)^{-1}(\lambda I - T)S(\lambda I - T)^{-1} = SR_T(\lambda)$ for $\lambda \in \rho(T)$.

Therefore, according to Definition 4.14, since S lies in $\mathcal{B}[\mathcal{X}]$,

$$S\psi(T) = S \frac{1}{2\pi i} \int_\Gamma \psi(\lambda)\, R_T(\lambda)\, d\lambda = \frac{1}{2\pi i} \int_\Gamma \psi(\lambda)\, SR_T(\lambda)\, d\lambda$$

$$= \frac{1}{2\pi i} \int_\Gamma \psi(\lambda)\, R_T(\lambda)S\, d\lambda = \psi(T)S. \qquad \square$$

Theorem 2.7 is the Spectral Mapping Theorem for polynomials, which holds for every Banach space operator. For normal operators on a Hilbert space, the Spectral Mapping Theorem was extended to larger classes of functions in Theorems 2.8, 4.11, and 4.12. Back to the general case of arbitrary Banach space operators, Theorem 2.7 can be extended to analytic functions, and this is again referred to as the SPECTRAL MAPPING THEOREM.

Theorem 4.18. *Take an operator $T \in \mathcal{B}[\mathcal{X}]$ on a complex Banach space $\mathcal{X}$. If ψ is analytic on the spectrum of T (i.e., if $\psi \in \mathcal{A}(\sigma(T))$), then*

$$\sigma(\psi(T)) = \psi(\sigma(T)).$$

Proof. Take $T \in \mathcal{B}[\mathcal{X}]$ and $\psi \colon \Lambda \to \mathbb{C}$ in $\mathcal{A}(\sigma(T))$. If $\lambda \in \sigma(T)$ (so that $\lambda \in \Lambda$), consider the function $\phi \colon \Lambda \to \mathbb{C}$ defined by $\phi(\lambda) = 0$ and $\phi(\nu) = \frac{\psi(\lambda) - \psi(\nu)}{\lambda - \nu}$ for every $\nu \in \Lambda \backslash \{\lambda\}$. Since the function $\psi(\lambda) - \psi(\cdot) = \phi(\cdot)(\lambda - \cdot) \colon \Lambda \to \mathbb{C}$ lies in $\mathcal{A}(\sigma(T))$, Lemma 4.15, Theorem 4.16, and Corollary 4.17 ensure that

$$\psi(\lambda)I - \psi(T) = \tfrac{1}{2\pi i} \int_\Gamma [\psi(\lambda) - \psi(\nu)] \, R_T(\nu) \, d\nu$$

$$= \tfrac{1}{2\pi i} \int_\Gamma (\lambda - \nu) \, \phi(\nu) \, R_T(\nu) \, d\nu$$

$$= (\lambda I - T) \, \phi(T) = \phi(T) \, (\lambda I - T).$$

If $\psi(\lambda) \in \rho(\psi(T))$, then $\psi(\lambda)I - \psi(T)$ has an inverse $[\psi(\lambda)I - \psi(T)]^{-1}$ in $\mathcal{B}[\mathcal{X}]$. Since $\psi(\lambda)I - \psi(T) = \phi(T) \, (\lambda I - T)$, we get

$$[\psi(\lambda)I - \psi(T)]^{-1} \phi(T) \, (\lambda I - T) = [\psi(\lambda)I - \psi(T)]^{-1}[\psi(\lambda)I - \psi(T)] = I,$$

$$(\lambda I - T) \, \phi(T) \, [\psi(\lambda)I - \psi(T)]^{-1} = [\psi(\lambda)I - \psi(T)] \, [\psi(\lambda)I - \psi(T)]^{-1} = I.$$

Thus $\lambda I - T$ has a left and a right bounded inverse, and so it has an inverse in $\mathcal{B}[\mathcal{X}]$, which means that $\lambda \in \rho(T)$, which is a contradiction. Therefore, if $\lambda \in \sigma(T)$, then $\psi(\lambda) \in \sigma(\psi(T))$, and hence

$$\psi(\sigma(T)) = \big\{ \psi(\lambda) \in \mathbb{C} \colon \lambda \in \sigma(T) \big\} \subseteq \sigma(\psi(T)).$$

Conversely, take any $\nu \notin \psi(\sigma(T))$, which means that $\nu - \psi(\lambda) \neq 0$ for every $\lambda \in \sigma(T)$, and consider the function $\phi'(\cdot) = \frac{1}{\nu - \psi(\cdot)} \colon \Lambda' \to \mathbb{C}$, which lies in $A(\sigma(T))$ with $\Lambda' \subseteq \Lambda$. Take Γ such that $\sigma(T) \subset \text{ins } \Gamma \subseteq (\text{ins } \Gamma)^- \subset \Lambda'$ as in Definition 4.14. Then Lemma 4.15 and Theorem 4.16 ensure that

$$\phi'(T) \, [\nu I - \psi(T)] = [\nu I - \psi(T)] \, \phi'(T) = \tfrac{1}{2\pi i} \int_\Gamma R_T(\lambda) \, d\lambda = I.$$

Thus $\nu I - \psi(T)$ has a bounded inverse, $\phi'(T) \in \mathcal{B}[\mathcal{X}]$, and so $\nu \in \rho(\psi(T))$. Equivalently, if $\nu \in \sigma(\psi(T))$, then $\nu \in \psi(\sigma(T))$. That is,

$$\sigma(\psi(T)) \subseteq \psi(\sigma(T)). \qquad \square$$

4.4 Riesz Decomposition Theorem and Riesz Idempotents

A *clopen set* in a topological space is a set that is both open and closed in it. If a topological space has a nontrivial (i.e., proper and nonempty) clopen set, then (by definition) it is *disconnected*. Consider an operator $T \in \mathcal{B}[\mathcal{X}]$ on a complex Banach space $\mathcal{X}$. There are some different definitions of *spectral set*. We stick to the classical one [39, Definition VII.3.17]: A *spectral set* is a clopen set in the spectrum $\sigma(T)$ (i.e., a subset of $\sigma(T) \subseteq \mathbb{C}$ that is both open

and closed relative to $\sigma(T)$). Suppose $\Delta \subseteq \sigma(T)$ is a spectral set. Then there is a function $\psi_\Delta \in A(\sigma(T))$ (analytic on a neighborhood of $\sigma(T)$) such that

$$\psi_\Delta(\lambda) = \begin{cases} 1, & \lambda \in \Delta, \\ 0, & \lambda \in \sigma(T) \backslash \Delta. \end{cases}$$

It does not matter which value $\psi_\Delta(\lambda)$ takes for those λ in a neighborhood of $\sigma(T)$ but not in $\sigma(T)$. Recall that if Δ is nontrivial (i.e., if $\varnothing \neq \Delta \neq \sigma(T)$), then $\sigma(T)$ must be disconnected. Set $E_\Delta = E_\Delta(T) = \psi_\Delta(T)$ so that

$$E_\Delta = \psi_\Delta(T) = \tfrac{1}{2\pi i} \int_\Gamma \psi_\Delta(\lambda) \, R_T(\lambda) \, d\lambda = \tfrac{1}{2\pi i} \int_{\Gamma_\Delta} R_T(\lambda) \, d\lambda$$

for any path Γ such that $\sigma(T) \subset$ ins Γ and (ins Γ)$^-$ is included in a neighborhood of $\sigma(T)$ as in Definition 4.14, and Γ_Δ is any path for which $\Delta \subseteq$ ins Γ_Δ and $\sigma(T) \backslash \Delta \subseteq$ out Γ_Δ. The operator $E_\Delta \in \mathcal{B}[\mathcal{X}]$ is referred to as the *Riesz idempotent* associated with Δ. In particular, the Riesz idempotent associated with an isolated point λ_0 of the spectrum of T, namely, $E_{\{\lambda_0\}} = E_{\{\lambda_0\}}(T) = \psi_{\{\lambda_0\}}(T)$, will be denoted by

$$E_{\lambda_0} = \tfrac{1}{2\pi i} \int_{\Gamma_{\lambda_0}} R_T(\lambda) \, d\lambda = \tfrac{1}{2\pi i} \int_{\Gamma_{\lambda_0}} (\lambda I - T)^{-1} \, d\lambda,$$

where Γ_{λ_0} is any simple closed rectifiable positively oriented curve (e.g., any positively oriented circle) enclosing λ_0 but no other point of $\sigma(T)$.

Let T be an operator in $\mathcal{B}[\mathcal{X}]$. It is readily verified that the collection of all spectral subsets of $\sigma(T)$ forms a (Boolean) algebra of subsets of $\sigma(T)$. Indeed, the empty set and whole set $\sigma(T)$ are spectral sets, complements relative to $\sigma(T)$ of spectral sets remain spectral sets, and finite unions of spectral sets are spectral sets, and so are differences and intersections of spectral sets.

Lemma 4.19. *Take any operator $T \in \mathcal{B}[\mathcal{X}]$ and let $E_\Delta \in \mathcal{B}[\mathcal{X}]$ be the Riesz idempotent associated with an arbitrary spectral subset Δ of $\sigma(T)$.*

(a) *E_Δ is a projection such that*

   (a$_1$) *$E_\varnothing = O$, $E_{\sigma(T)} = I$ and $E_{\sigma(T) \backslash \Delta} = I - E_\Delta$.*

   If Δ_1 and Δ_2 are spectral subsets of $\sigma(T)$, then

   (a$_2$) *$E_{\Delta_1 \cap \Delta_2} = E_{\Delta_1} E_{\Delta_2}$ and $E_{\Delta_1 \cup \Delta_2} = E_{\Delta_1} + E_{\Delta_2} - E_{\Delta_1} E_{\Delta_2}$.*

(b) *E_Δ commutes with every operator that commutes with T.*

(c) *$\mathcal{R}(E_\Delta)$ is a subspace of $\mathcal{X}$ that is S-invariant for every operator $S \in \mathcal{B}[\mathcal{X}]$ that commutes with T.*

Proof. (a) Lemma 4.15 ensures that a Riesz idempotent deserves its name: $E_\Delta = E_\Delta^2$ so that $E_\Delta \in \mathcal{B}[\mathcal{X}]$ is a projection. Observe from the definition of the function ψ_Δ for an arbitrary spectral subset Δ of $\sigma(T)$ that

(i) $\psi_\varnothing = 0, \quad \psi_{\sigma(T)} = 1, \quad \psi_{\sigma(T)\setminus\Delta} = 1 - \psi_\Delta,$

(ii) $\psi_{\Delta_1 \cap \Delta_2} = \psi_{\Delta_1}\psi_{\Delta_2} \quad$ and $\quad \psi_{\Delta_1 \cup \Delta_2} = \psi_{\Delta_1} + \psi_{\Delta_2} - \psi_{\Delta_1}\psi_{\Delta_2}.$

Since $E_\Delta = \psi_\Delta(T) = \frac{1}{2\pi i}\int_\Gamma \psi_\Delta(\lambda)\,R_T(\lambda)\,d\lambda$ for every spectral subset Δ of $\sigma(T)$, where Γ is any path as in Definition 4.14, the identities in (a$_1$) and (a$_2$) follow from the identities in (i) and (ii), respectively (using the linearity of the integral, Lemma 4.15, and Theorem 4.16).

(b) Suppose $S \in \mathcal{B}[\mathcal{X}]$ commutes with T. Corollary 4.17 ensures that

$$ST = TS \quad \text{implies} \quad SE_\Delta = E_\Delta S.$$

(c) Thus $\mathcal{R}(E_\Delta)$ is S-invariant. Moreover $\mathcal{R}(E_\Delta)$ is a subspace (i.e., a *closed* linear manifold) of $\mathcal{X}$ since E_Δ is bounded and $\mathcal{R}(E_\Delta) = \mathcal{N}(I - E_\Delta)$. In particular, $\mathcal{R}(E_\Delta)$ is an invariant subspace for T. $\square$

Theorem 4.20. *Take $T \in \mathcal{B}[\mathcal{X}]$. If Δ is any spectral subset of $\sigma(T)$, then*

$$\sigma(T|_{\mathcal{R}(E_\Delta)}) = \Delta.$$

Moreover, the map $\Delta \mapsto E_\Delta$ that assigns to each spectral subset of $\sigma(T)$ the associated Riesz idempotent $E_\Delta \in \mathcal{B}[\mathcal{X}]$ is injective.

Proof. Consider the results in Lemma 4.19(a$_1$). If $\Delta = \sigma(T)$, then $E_\Delta = I$ so that $\mathcal{R}(E_\Delta) = \mathcal{X}$, and hence $\sigma(T|_{\mathcal{R}(E_\Delta)}) = \sigma(T)$. On the other hand, if $\Delta = \varnothing$, then $E_\Delta = O$ so that $\mathcal{R}(E_\Delta) = \{0\}$, and hence $T|_{\mathcal{R}(E_\Delta)}: \{0\} \to \{0\}$ is the null operator on the zero space, which implies that $\sigma(T|_{\mathcal{R}(E_\Delta)}) = \varnothing$. In both cases the identity $\sigma(T|_{\mathcal{R}(E_\Delta)}) = \Delta$ holds trivially. (Recall that the spectrum of a *bounded* linear operator is nonempty on every *nonzero* complex Banach space.) Thus suppose $\varnothing \neq \Delta \neq \sigma(T)$. Let $\psi_\Delta \in A(\sigma(T))$ be the function that defines the Riesz idempotent E_Δ associated with Δ,

$$\psi_\Delta(\lambda) = \begin{cases} 1, & \lambda \in \Delta, \\ 0, & \lambda \in \sigma(T)\setminus\Delta, \end{cases}$$

so that

$$E_\Delta = \psi_\Delta(T) = \frac{1}{2\pi i}\int_\Gamma \psi_\Delta(\lambda)\,R_T(\lambda)\,d\lambda$$

for any path Γ as in Definition 4.14, and let $\varphi \in A(\sigma(T))$ be the identity function on $\sigma(T)$, that is,

$$\varphi(\lambda) = \lambda \quad \text{for every} \quad \lambda \in \sigma(T),$$

so that (cf. Theorem 4.16)

$$T = \varphi(T) = \frac{1}{2\pi i}\int_\Gamma \varphi(\lambda)\,R_T(\lambda)\,d\lambda.$$

Claim 1. $\sigma(TE_\Delta) = \Delta \cup \{0\}.$

Proof. Since $TE_\Delta = \varphi(T)\psi_\Delta(T)$ and

$$\varphi(T)\psi_\Delta(T) = \tfrac{1}{2\pi i} \int_\Gamma \varphi(\lambda)\psi_\Delta(\lambda) R_T(\lambda)\,d\lambda = \tfrac{1}{2\pi i} \int_\Gamma (\varphi\,\psi_\Delta)(\lambda) R_T(\lambda)\,d\lambda,$$

and since $\varphi\,\psi_\Delta \in A(\sigma(T))$, the Spectral Mapping Theorem ensures that

$$\sigma(TE_\Delta) = (\varphi\,\psi_\Delta)(\sigma(T)) = \varphi(\sigma(T))\,\psi_\Delta(\sigma(T)) = \sigma(T)\,\psi_\Delta(\sigma(T)) = \Delta \cup \{0\}$$

(cf. Lemma 4.15 and Theorems 4.16 and 4.18).

Claim 2. $\sigma(TE_\Delta) = \sigma(T|_{\mathcal{R}(E_\Delta)}) \cup \{0\}$.

Proof. Since E_Δ is a projection, $\mathcal{R}(E_\Delta)$ and $\mathcal{N}(E_\Delta)$ are algebraic complements, which means that $\mathcal{R}(E_\Delta) \cap \mathcal{N}(E_\Delta) = \varnothing$ and $\mathcal{X} = \mathcal{R}(E_\Delta) \oplus \mathcal{N}(E_\Delta)$ — plain direct sum (no orthogonality in a pure Banach space setup). Thus, with respect to the complementary decomposition $\mathcal{X} = \mathcal{R}(E_\Delta) \oplus \mathcal{N}(E_\Delta)$, we get $E_\Delta = \left(\begin{smallmatrix} I & O \\ O & O \end{smallmatrix}\right) = I \oplus O$, and so $TE_\Delta = \left(\begin{smallmatrix} T|_{\mathcal{R}(E_\Delta)} & O \\ O & O \end{smallmatrix}\right) = T|_{\mathcal{R}(E_\Delta)} \oplus O$ because $E_\Delta T = TE_\Delta$. Then $\lambda I - TE_\Delta = (\lambda I - T|_{\mathcal{R}(E_\Delta)}) \oplus \lambda I$ for every $\lambda \in \mathbb{C}$, and hence $\lambda \in \rho(E_\Delta T)$ if and only if $\lambda \in \rho(T|_{\mathcal{R}(E_\Delta)})$ and $\lambda \neq 0$. Equivalently, $\lambda \in \sigma(E_\Delta T)$ if and only if $\lambda \in \sigma(T|_{\mathcal{R}(E_\Delta)})$ or $\lambda = 0$.

Claim 3. $\sigma(T|_{\mathcal{R}(E_\Delta)}) \cup \{0\} = \Delta \cup \{0\}$.

Proof. Claims 1 and 2.

Claim 4. $0 \in \Delta \iff 0 \in \sigma(T|_{\mathcal{R}(E_\Delta)})$.

Proof. Suppose $0 \in \Delta$. Thus there exists a function $\phi_0 \in A(\sigma(T))$ such that

$$\phi_0(\lambda) = \begin{cases} 0, & \lambda \in \Delta, \\ \lambda^{-1}, & \lambda \in \sigma(T)\backslash\Delta. \end{cases}$$

Observe that $\varphi\,\phi_0$ lies in $A(\sigma(T))$ and

$$\varphi\,\phi_0 = 1 - \psi_\Delta \quad \text{on} \quad \sigma(T),$$

and hence, since $\varphi(T) = T$ and $\phi_0(T) \in \mathcal{B}[\mathcal{X}]$,

$$T\phi_0(T) = \phi_0(T)T = I - \psi_\Delta(T) = I - E_\Delta.$$

If $0 \notin \sigma(T|_{\mathcal{R}(E_\Delta)})$, then $T|_{\mathcal{R}(E_\Delta)}$ has a bounded inverse on the Banach space $\mathcal{R}(E_\Delta)$, which means that there exists an operator $[T|_{\mathcal{R}(E_\Delta)}]^{-1}$ on $\mathcal{R}(E_\Delta)$ such that $T|_{\mathcal{R}(E_\Delta)}[T|_{\mathcal{R}(E_\Delta)}]^{-1} = [T|_{\mathcal{R}(E_\Delta)}]^{-1}T|_{\mathcal{R}(E_\Delta)} = I$. Consider the operator $[T|_{\mathcal{R}(E_\Delta)}]^{-1}E_\Delta$ in $\mathcal{B}[\mathcal{X}]$ so that, since $T\psi_\Delta(T) = \psi_\Delta(T)T$,

$$T\Big([T|_{\mathcal{R}(E_\Delta)}]^{-1}E_\Delta + \phi_0(T)\Big) = \Big([T|_{\mathcal{R}(E_\Delta)}]^{-1}E_\Delta + \phi_0(T)\Big)T = I \in \mathcal{B}[\mathcal{X}].$$

Then T has a bounded inverse on $\mathcal{X}$, namely, $[T|_{\mathcal{R}(E_\Delta)}]^{-1}E_\Delta + \phi_0(T)$, so that $0 \in \rho(T)$, which is a contradiction (since $0 \in \Delta \subseteq \sigma(T)$). Therefore,

$$0 \in \Delta \quad \text{implies} \quad 0 \in \sigma(T|_{\mathcal{R}(E_\Delta)}).$$

Conversely, suppose $0 \notin \Delta$. Thus there is a function $\phi_1 \in A(\sigma(T))$ such that

$$\phi_1(\lambda) = \begin{cases} \lambda^{-1}, & \lambda \in \Delta, \\ 0, & \lambda \in \sigma(T) \backslash \Delta. \end{cases}$$

Again, observe that $\varphi \, \phi_1$ lies in $A(\sigma(T))$ and

$$\varphi \, \phi_1 = \psi_\Delta \quad \text{on} \quad \sigma(T),$$

and hence, since $\varphi(T) = T$ and $\phi_1(T) \in \mathcal{B}[\mathcal{X}]$,

$$T\phi_1(T) = \phi_1(T)T = \psi_\Delta(T) = E_\Delta.$$

Since $\phi_1(T)\psi_\Delta(T) = \psi_\Delta(T)\phi_1(T)$ by Lemma 4.15, it follows that the operator $\phi_1(T)$ is $\mathcal{R}(E_\Delta)$-invariant. So $\phi_1(T)|_{\mathcal{R}(E_\Delta)}$ lies in $\mathcal{B}[\mathcal{R}(E_\Delta)]$, and is such that

$$T|_{\mathcal{R}(E_\Delta)}\phi_1(T)|_{\mathcal{R}(E_\Delta)} = \phi_1(T)|_{\mathcal{R}(E_\Delta)}T|_{\mathcal{R}(E_\Delta)} = E_\Delta|_{\mathcal{R}(E_\Delta)} = I \ \in \ \mathcal{B}[\mathcal{R}(E_\Delta)].$$

Then $T|_{\mathcal{R}(E_\Delta)}$ has a bounded inverse on $\mathcal{R}(E_\Delta)$, namely, $\phi_1(T)|_{\mathcal{R}(E_\Delta)}$, so that $0 \in \rho(T|_{\mathcal{R}(E_\Delta)})$. Thus $0 \notin \Delta$ implies $0 \notin \sigma(T|_{\mathcal{R}(E_\Delta)})$. Equivalently,

$$0 \in \sigma(T|_{\mathcal{R}(E_\Delta)}) \quad \text{implies} \quad 0 \in \Delta.$$

This concludes the proof of Claim 4.

Claims 3 and 4 ensure the identity

$$\sigma(T|_{\mathcal{R}(E_\Delta)}) = \Delta.$$

Finally consider the map $\Delta \mapsto E_\Delta$ that assigns to each spectral subset of $\sigma(T)$ the associated Riesz idempotent $E_\Delta \in \mathcal{B}[\mathcal{X}]$. We have already seen at the beginning of this proof that if $E_\Delta = O$, then $\sigma(T|_{\mathcal{R}(E_\Delta)}) = \varnothing$. Hence

$$E_\Delta = O \quad \Longrightarrow \quad \Delta = \varnothing$$

by the above displayed identity (that holds for all spectral subsets of $\sigma(T)$). Take arbitrary spectral subsets Δ_1 and Δ_2 of $\sigma(T)$. From Lemma 4.19(a),

$$E_{\Delta_1 \backslash \Delta_2} = E_{\Delta_1 \cap (\sigma(T) \backslash \Delta_2)} = E_{\Delta_1}(I - E_{\Delta_2}) = E_{\Delta_1} - E_{\Delta_1}E_{\Delta_2},$$

where E_{Δ_1} and E_{Δ_2} commute since $E_{\Delta_1}E_{\Delta_2} = E_{\Delta_1 \cap \Delta_2} = E_{\Delta_2}E_{\Delta_1}$, and so

$$(E_{\Delta_1} - E_{\Delta_2})^2 = E_{\Delta_1} + E_{\Delta_2} - 2E_{\Delta_1}E_{\Delta_2} = E_{\Delta_1 \backslash \Delta_2} + E_{\Delta_2 \backslash \Delta_1} = E_{\Delta_1 \triangledown \Delta_2},$$

where $\Delta_1 \triangledown \Delta_2 = (\Delta_1 \backslash \Delta_2) \cup (\Delta_2 \backslash \Delta_1)$ is the symmetric difference. Thus

$$E_{\Delta_1} = E_{\Delta_2} \quad \Longrightarrow \quad E_{\Delta_1 \triangledown \Delta_2} = O \quad \Longrightarrow \quad \Delta_1 \triangledown \Delta_2 = \varnothing \quad \Longrightarrow \quad \Delta_1 = \Delta_2,$$

since $\Delta_1 \triangledown \Delta_2 = \varnothing$ if and only if $\Delta_1 = \Delta_2$, which proves injectivity. $\qquad\square$

Remark. Let T be an operator in $\mathcal{B}[\mathcal{X}]$ and suppose $\sigma(T)$ is disconnected, so that there are nontrivial sets in the algebra of all spectral subsets of $\sigma(T)$. We show that to each partition of $\sigma(T)$ into spectral sets there is a corresponding decomposition of the resolvent function R_T. Indeed, consider a finite partition $\{\Delta_j\}_{j=1}^n$ of nontrivial spectral subsets of $\sigma(T)$; that is,

$$\varnothing \neq \Delta_j \neq \sigma(T) \text{ are clopen subsets of } \sigma(T) \text{ for each } j = 1, \ldots, n$$

such that

$$\bigcup_{j=1}^n \Delta_j = \sigma(T) \quad \text{with} \quad \Delta_j \cap \Delta_k = \varnothing \text{ whenever } j \neq k.$$

Consider the Riesz idempotent associated with each Δ_j, namely,

$$E_{\Delta_j} = \tfrac{1}{2\pi i} \int_{\Gamma_{\Delta_j}} R_T(\lambda) \, d\lambda.$$

The results of Lemma 4.19 are readily extended for any integer $n \geq 2$ so that each E_{Δ_j} is a nontrivial projection $(O \neq E_{\Delta_j} = E_{\Delta_j}^2 \neq I)$ and

$$\sum_{j=1}^n E_{\Delta_j} = I \quad \text{with} \quad E_{\Delta_j} E_{\Delta_k} = O \text{ whenever } j \neq k.$$

Take an arbitrary $\lambda \in \rho(T)$. For each $j = 1, \ldots, n$ set

$$T_j = T E_{\Delta_j} \quad \text{and} \quad R_j(\lambda) = R_T(\lambda) E_{\Delta_j}$$

in $\mathcal{B}[\mathcal{X}]$ so that

$$\sum_{j=1}^n T_j = T \quad \text{and} \quad \sum_{j=1}^n R_j(\lambda) = R_T(\lambda).$$

Observe that these operators commute. Indeed, recall that $T R_T(\lambda) = R_T(\lambda) T$, and $T E_{\Delta_j} = E_{\Delta_j} T$, and hence $R_T(\lambda) E_{\Delta_j} = E_{\Delta_j} R_T(\lambda)$ (cf. Corollary 4.17 and Lemma 4.19). Therefore, if $j \neq k$, then

$$T_j T_k = R_j(\lambda) R_k(\lambda) = T_k R_j(\lambda) = R_j(\lambda) T_k = O.$$

On the other hand, for every $j = 1, \ldots, n$,

$$T_j R_j(\lambda) = R_j(\lambda) T_j = T R_T(\lambda) E_{\Delta_j}$$

and, since $(\lambda I - T) R_T(\lambda) = R_T(\lambda)(\lambda I - T) = I$,

$$\big(\lambda E_{\Delta_j} - T_j\big) R_j(\lambda) = R_j(\lambda)\big(\lambda E_{\Delta_j} - T_j\big) = E_{\Delta_j}.$$

Moreover, Claim 1 in the proof of Theorem 4.20 says that

$$\sigma(T_j) = \Delta_j \cup \{0\}.$$

Note that the spectral radii are such that $r(T_j) \leq r(T)$. Since $T_j^k = T^k E_{\Delta_j}$ for every nonnegative integer k, it follows by Corollary 2.12 that

$$R_j(\lambda) = (\lambda I - T)^{-1} E_{\Delta_j} = \tfrac{1}{\lambda} \sum_{k=0}^{\infty} \left(\tfrac{T}{\lambda}\right)^k E_{\Delta_j} = \tfrac{1}{\lambda} \sum_{k=0}^{\infty} \left(\tfrac{T_j}{\lambda}\right)^k$$

if λ in $\rho(T)$ is such that $r(T) < |\lambda|$ and, again by Corollary 2.12,

$$R_{T_j}(\lambda) = (\lambda I - T E_{\Delta_j})^{-1} = \tfrac{1}{\lambda} \sum_{k=0}^{\infty} \left(\tfrac{T E_{\Delta_j}}{\lambda}\right)^k = \tfrac{1}{\lambda} \sum_{k=0}^{\infty} \left(\tfrac{T_j}{\lambda}\right)^k$$

if λ in $\rho(T_j)$ is such that $r(T_j) < |\lambda|$, where the above series converge uniformly in $\mathcal{B}[\mathcal{X}]$. Hence, if $r(T) < |\lambda|$, then

$$R_j(\lambda) = R_T(\lambda) E_{\Delta_j} = R_{T \Delta_j} = R_{T_j}$$

where each R_j can actually be extended to be analytic in $\rho(T_j)$.

The properties of Riesz idempotents in Lemma 4.19 and Theorem 4.20 lead to a major result in operator theory, which says that *operators whose spectra are disconnected have nontrivial hyperinvariant subspaces*. (For the original statement of the Riesz Decomposition Theorem see [77, p. 421].)

Corollary 4.21. (RIESZ DECOMPOSITION THEOREM). *Let $T \in \mathcal{B}[\mathcal{X}]$ be an operator on a complex Banach space $\mathcal{X}$. If $\sigma(T) = \Delta_1 \cup \Delta_2$, where Δ_1 and Δ_2 are disjoint nonempty closed sets, then T has a complementary pair $\{\mathcal{M}_1, \mathcal{M}_2\}$ of nontrivial hyperinvariant subspaces, viz., $\mathcal{M}_1 = \mathcal{R}(E_{\Delta_1})$ and $\mathcal{M}_2 = \mathcal{R}(E_{\Delta_2})$, such that $\sigma(T|_{\mathcal{M}_1}) = \Delta_1$ and $\sigma(T|_{\mathcal{M}_2}) = \Delta_2$.*

Proof. If Δ_1 and Δ_2 are disjoint closed sets in $\mathbb{C}$ such that $\sigma(T) = \Delta_1 \cup \Delta_2$, then they are both clopen subsets of $\sigma(T)$ (i.e., spectral subsets of $\sigma(T)$), and hence the Riesz idempotents E_{Δ_1} and E_{Δ_2} associated with them are such that their ranges $\mathcal{M}_1 = \mathcal{R}(E_{\Delta_1})$ and $\mathcal{M}_2 = \mathcal{R}(E_{\Delta_2})$ are subspaces of $\mathcal{X}$ that are hyperinvariant for T by Lemma 4.19(c). Since Δ_1 and Δ_2 are nontrivial (i.e., $\varnothing \neq \Delta_1 \neq \sigma(T)$ and $\varnothing \neq \Delta_2 \neq \sigma(T)$), it follows that $\sigma(T)$ is disconnected (since they are both clopen subsets of $\sigma(T)$) and the projections E_{Δ_1} and E_{Δ_2} are nontrivial (i.e., $O \neq E_{\Delta_1} \neq I$ and $O \neq E_{\Delta_2} \neq I$ by the injectivity of Theorem 4.20), and so the subspaces $\mathcal{M}_1$ and $\mathcal{M}_2$ are nontrivial (i.e., $\{0\} \neq \mathcal{R}(E_{\Delta_1}) \neq \mathcal{X}$ and $\{0\} \neq \mathcal{R}(E_{\Delta_2}) \neq \mathcal{X}$). Lemma 4.19(a) ensures that $E_{\Delta_1} + E_{\Delta_2} = I$ and $E_{\Delta_1} E_{\Delta_2} = O$, which means that E_{Δ_1} and E_{Δ_2} are complementary projections (not necessarily orthogonal even if $\mathcal{X}$ were a Hilbert space), and so their ranges are complementary subspaces (i.e., $\mathcal{M}_1 + \mathcal{M}_2 = \mathcal{X}$ and $\mathcal{M}_1 \cap \mathcal{M}_2 = \{0\}$) as in Section 1.4. Finally, Theorem 4.20 ensures that $\sigma(T|_{\mathcal{M}_1}) = \Delta_1$ and $\sigma(T|_{\mathcal{M}_2}) = \Delta_2$. $\qquad \square$

Remark. Since $\mathcal{M}_1 = \mathcal{R}(E_{\Delta_1})$ and $\mathcal{M}_2 = \mathcal{R}(E_{\Delta_2})$ are complementary subspaces (i.e., $\mathcal{M}_1 + \mathcal{M}_2 = \mathcal{X}$ and $\mathcal{M}_1 \cap \mathcal{M}_2 = \{0\}$), it follows that $\mathcal{X}$ can be identified with the direct sum $\mathcal{M}_1 \oplus \mathcal{M}_2$ (not necessarily an orthogonal direct sum even if $\mathcal{X}$ were a Hilbert space), which means that there exists a

natural isomorphism $\Psi \colon \mathcal{X} \to \mathcal{M}_1 \oplus \mathcal{M}_2$ between the normed spaces $\mathcal{X}$ and $\mathcal{M}_1 \oplus \mathcal{M}_2$ (see, e.g., [66, Theorem 2.14]). The normed space $\mathcal{M}_1 \oplus \mathcal{M}_2$ is not necessarily complete and the invertible linear transformation Ψ is not necessarily bounded. In other words, $\mathcal{X} \sim \mathcal{M}_1 \oplus \mathcal{M}_2$, where $\sim$ stands for algebraic similarity (i.e., isomorphic equivalence) and $\oplus$ for direct (not necessarily orthogonal) sum. Thus, since $\mathcal{M}_1$ and $\mathcal{M}_2$ are both T-invariant (recall that both E_{Δ_1} and E_{Δ_2} commute with T by Lemma 4.19), it follows that $T \sim T|_{\mathcal{M}_1} \oplus T|_{\mathcal{M}_2}$, which means $\Psi T \Psi^{-1} = T|_{\mathcal{M}_1} \oplus T|_{\mathcal{M}_2}$. (If $\mathcal{X}$ were Hilbert and the subspaces orthogonal, then we might say that they reduce T.)

We say that a pair Δ_1 and Δ_2 of subsets of $\sigma(T)$ are *complementary spectral sets* for T if, besides being spectral sets (i.e., subsets of $\sigma(T)$ that are both open and closed relative to $\sigma(T)$) they also form a nontrivial partition of $\sigma(T)$; that is, $\varnothing \neq \Delta_1 \neq \sigma(T)$, $\varnothing \neq \Delta_2 \neq \sigma(T)$, $\Delta_1 \cup \Delta_2 = \sigma(T)$, and $\Delta_1 \cap \Delta_2 = \varnothing$. Observe that the Riesz Decomposition Theorem says that, *for every pair of complementary spectral sets, the ranges of their Riesz idempotents are complementary nontrivial hyperinvariant subspaces for T, and the spectra of the restrictions of T to those ranges coincide with themselves.*

Corollary 4.22. *Let Δ_1 and Δ_2 be complementary spectral sets for an operator $T \in \mathcal{B}[\mathcal{X}]$, and let E_{Δ_1} and E_{Δ_2} be the Riesz idempotents associated with them. If $\lambda \in \Delta_1$, then*

$$E_{\Delta_2}(\mathcal{N}(\lambda I - T)) = \{0\},$$

$$E_{\Delta_2}(\mathcal{R}(\lambda I - T)) = \mathcal{R}(E_{\Delta_2}),$$

$$E_{\Delta_1}(\mathcal{N}(\lambda I - T)) = \mathcal{N}(\lambda I - T) \subseteq \mathcal{R}(E_{\Delta_1}),$$

$$E_{\Delta_1}(\mathcal{R}(\lambda I - T)) = \mathcal{R}((\lambda I - T)E_{\Delta_1}) \subseteq \mathcal{R}(E_{\Delta_1}).$$

so that

$$\mathcal{R}(\lambda I - T) = E_{\Delta_1}(\mathcal{R}(\lambda I - T)) + \mathcal{R}(E_{\Delta_2}).$$

If $\dim \mathcal{R}(E_{\Delta_1}) < \infty$, then $\dim \mathcal{N}(\lambda I - T) < \infty$ and $\mathcal{R}(\lambda I - T)$ is closed.

Proof. Let T be an operator on a complex Banach space $\mathcal{X}$.

Claim 1. If Δ_1 and Δ_2 are complementary spectral sets and $\lambda \in \mathbb{C}$, then

(a) $\qquad \mathcal{N}(\lambda I - T) = E_{\Delta_1}(\mathcal{N}(\lambda I - T)) + E_{\Delta_2}(\mathcal{N}(\lambda I - T)),$

(b) $\qquad \mathcal{R}(\lambda I - T) = E_{\Delta_1}(\mathcal{R}(\lambda I - T)) + E_{\Delta_2}(\mathcal{R}(\lambda I - T)).$

Proof. Let E_{Δ_1} and E_{Δ_2} be the Riesz idempotents associated with Δ_1 and Δ_2. Lemma 4.19(a) ensures that $\mathcal{R}(E_{\Delta_1})$ and $\mathcal{R}(E_{\Delta_2})$ are complementary subspaces (i.e., $\mathcal{X} = \mathcal{R}(E_{\Delta_1}) + \mathcal{R}(E_{\Delta_2})$ and $\mathcal{R}(E_{\Delta_1}) \cap \mathcal{R}(E_{\Delta_2}) = \varnothing$, since Δ_1 and Δ_2 are complementary spectral sets, so that E_{Δ_1} and E_{Δ_2} are complementary projections). Thus (cf. Section 1.1), there is a unique decomposition

$$x = u + v = E_{\Delta_1} x + E_{\Delta_2} x$$

for every vector $x \in \mathcal{X}$, with $u \in \mathcal{R}(E_{\Delta_1})$ and $v \in \mathcal{R}(E_{\Delta_2})$, where $u = E_{\Delta_1} x$ and $v = E_{\Delta_2} x$ (since $E_{\Delta_1}(\mathcal{R}(E_{\Delta_2})) = \{0\}$ and $E_{\Delta_2}(\mathcal{R}(E_{\Delta_1})) = \{0\}$ because $E_{\Delta_1} E_{\Delta_2} = E_{\Delta_2} E_{\Delta_1} = O$). So we get the decompositions in (a) and (b).

Claim 2. If $\lambda \in \sigma(T) \backslash \Delta$ for some spectral set Δ of $\sigma(T)$, then

$$E_\Delta(\mathcal{N}(\lambda I - T)) = \{0\} \quad \text{and} \quad E_\Delta(\mathcal{R}(\lambda I - T)) = \mathcal{R}(E_\Delta).$$

Proof. Let Δ be any spectral set of $\sigma(T)$ and let E_Δ be the Riesz idempotent associated to it. Since $\mathcal{R}(E_\Delta)$ is an invariant subspace for T (cf. Lemma 4.19), take the restriction $T|_{\mathcal{R}(E_\Delta)} \in \mathcal{B}[\mathcal{R}(E_\Delta)]$ of T to the Banach space $\mathcal{R}(E_\Delta)$. If $\lambda \in \sigma(T) \backslash \Delta$, then $\lambda \in \rho(T|_{\mathcal{R}(E_\Delta)})$. So $(\lambda I - T)|_{\mathcal{R}(E_\Delta)} = \lambda I|_{\mathcal{R}(E_\Delta)} - T|_{\mathcal{R}(E_\Delta)}$ has a bounded inverse, where $I|_{\mathcal{R}(E_\Delta)} = E_\Delta|_{\mathcal{R}(E_\Delta)}$ stands for the identity on $\mathcal{R}(E_\Delta)$. Thus, since $\mathcal{R}(E_\Delta)$ is T-invariant,

$$
\begin{aligned}
R_{T|_{\mathcal{R}(E_\Delta)}}&(\lambda)\,(\lambda I - T)E_\Delta \\
&= (\lambda I|_{\mathcal{R}(E_\Delta)} - T|_{\mathcal{R}(E_\Delta)})^{-1}(\lambda I|_{\mathcal{R}(E_\Delta)} - T|_{\mathcal{R}(E_\Delta)})E_\Delta \\
&= I|_{\mathcal{R}(E_\Delta)} E_\Delta = E_\Delta,
\end{aligned}
$$

where $R_{T|_{\mathcal{R}(E_\Delta)}} \colon \rho(T|_{\mathcal{R}(E_\Delta)}) \to \mathcal{G}[\mathcal{R}(E_\Delta)]$ is the resolvent function of $T|_{\mathcal{R}(E_\Delta)}$. Now take an arbitrary $x \in \mathcal{N}(\lambda I - T)$. Since $E_\Delta T = T E_\Delta$,

$$E_\Delta x = R_{T|_{\mathcal{R}(E_\Delta)}}(\lambda)\,(\lambda I - T)E_\Delta x = R_{T|_{\mathcal{R}(E_\Delta)}}(\lambda)\,E_\Delta(\lambda I - T)x = 0.$$

Therefore,

$$E_\Delta(\mathcal{N}(\lambda I - T)) = \{0\}.$$

Moreover, since

$$(\lambda I - T)E_\Delta = \lambda E_\Delta - T E_\Delta = \lambda I|_{\mathcal{R}(E_\Delta)} - T|_{\mathcal{R}(E_\Delta)},$$

it follows that, if $\lambda \in \sigma(T) \backslash \Delta$, then $\lambda \in \rho(T|_{\mathcal{R}(E_\Delta)})$, by Theorem 4.20 (where $T|_{\mathcal{R}(E_\Delta)} \in \mathcal{B}[\mathcal{R}(E_\Delta)]$), so that (see diagram of Section 2.2)

$$\mathcal{R}((\lambda I - T)E_\Delta) = \mathcal{R}(\lambda I|_{\mathcal{R}(E_\Delta)} - T|_{\mathcal{R}(E_\Delta)}) = \mathcal{R}(E_\Delta).$$

Since $T E_\Delta = E_\Delta T$, and $A(\mathcal{R}(B)) = \mathcal{R}(AB)$ for all operators A and B,

$$E_\Delta(\mathcal{R}(\lambda I - T)) = \mathcal{R}(E_\Delta(\lambda I - T)) = \mathcal{R}((\lambda I - T)E_\Delta) = \mathcal{R}(E_\Delta).$$

This concludes the proof of Claim 2.

From now on suppose $\lambda \in \Delta_1$. In this case, Claim 2 ensures that

$$E_{\Delta_2}(\mathcal{N}(\lambda I - T)) = \{0\},$$

and so it follows by Claim 1(a) that

$$\mathcal{N}(\lambda I - T) = E_{\Delta_1}(\mathcal{N}(\lambda I - T)) \subseteq \mathcal{R}(E_{\Delta_1}).$$

Hence

$$\dim \mathcal{R}(E_{\Delta_1}) < \infty \quad \Longrightarrow \quad \dim(\mathcal{N}(\lambda I - T)) < \infty.$$

Again, from Claim 2 we get

$$E_{\Delta_2}(\mathcal{R}(\lambda I - T)) = \mathcal{R}(E_{\Delta_2}).$$

Thus, from Claim 1(b),

$$\mathcal{R}(\lambda I - T) = E_{\Delta_1}(\mathcal{R}(\lambda I - T)) + \mathcal{R}(E_{\Delta_2}),$$

where (since $TE_\Delta = E_\Delta T$ and $A(\mathcal{R}(B)) = \mathcal{R}(AB)$ for all A and B),

$$E_{\Delta_1}(\mathcal{R}(\lambda I - T)) = \mathcal{R}(E_{\Delta_1}(\lambda I - T)) = \mathcal{R}((\lambda I - T)E_{\Delta_1}) \subseteq \mathcal{R}(E_{\Delta_1}).$$

Recall that $\mathcal{R}(E_{\Delta_2})$ is a subspace (i.e., a closed linear manifold) of $\mathcal{X}$ (Lemma 4.19). If $\mathcal{R}(E_{\Delta_1})$ is finite dimensional, then $E_{\Delta_1}(\mathcal{R}(\lambda I - T))$ is finite dimensional, and so $\mathcal{R}(\lambda I - T)$ is the sum of a finite-dimensional linear manifold and a closed linear manifold, which is closed (cf. Proposition 1.C). Hence,

$$\dim(\mathcal{R}(E_{\Delta_1})) < \infty \quad \Longrightarrow \quad \mathcal{R}(\lambda I - T) \text{ is closed.} \qquad \square$$

Remark. Let λ_0 be an isolated point of the spectrum of an operator T in $\mathcal{B}[\mathcal{X}]$. A particular case of the preceding corollary for $\Delta_1 = \{\lambda_0\}$ says that, if $\dim \mathcal{R}(E_{\lambda_0}) < \infty$, then $\dim \mathcal{N}(\lambda_0 I - T) < \infty$ and $\mathcal{R}(\lambda_0 I - T)$ is closed. We shall prove the converse in Theorem 5.19 (the next chapter):

$$\dim \mathcal{R}(E_{\lambda_0}) < \infty \iff \dim \mathcal{N}(\lambda_0 I - T) < \infty \text{ and } \mathcal{R}(\lambda_0 I - T) \text{ is closed.}$$

Also note that $\lambda_0 I - T|_{\mathcal{R}(E_{\lambda_0})}$ is a quasinilpotent operator on $\mathcal{R}(E_{\lambda_0})$. Indeed, $\sigma(T|_{\mathcal{R}(E_{\lambda_0})}) = \{\lambda_0\}$ according to Theorem 4.20, and so the Spectral Mapping Theorem (Theorem 2.7) ensures that

$$\sigma(\lambda_0 I - T|_{\mathcal{R}(E_{\lambda_0})}) = \{0\}.$$

Corollary 4.23. *Let $T \in \mathcal{B}_\infty[\mathcal{X}]$ be a compact operator. If $\lambda \in \sigma(T) \backslash \{0\}$, then λ is an isolated point of $\sigma(T)$ such that*

$$\dim \mathcal{R}(E_\lambda) < \infty,$$

where $E_\lambda \in \mathcal{B}[\mathcal{X}]$ is the Riesz idempotent associated with it, and

$$\{0\} \neq \mathcal{N}(\lambda I - T) \subseteq \mathcal{R}(E_\lambda) \subseteq \mathcal{N}((\lambda I - T)^n)$$

for some a positive integer n such that

$$\mathcal{R}(E_\lambda) \not\subseteq \mathcal{N}(\lambda I - T)^{n-1}.$$

Proof. Let T be a compact operator on a nonzero complex Banach space $\mathcal{X}$. We refer to the results of Section 2.6, which were proved on a Hilbert space but, as we had mentioned there, still hold on a Banach space. If $\lambda \in \sigma(T)\backslash\{0\}$, then λ is an isolated point of $\sigma(T)$ by Corollary 2.20. Take the restriction $T|_{\mathcal{R}(E_\lambda)} \in \mathcal{B}[\mathcal{R}(E_\lambda)]$ of T to the Banach space $\mathcal{R}(E_\Delta)$, which is again compact (Proposition 1.V). Recall from Theorem 4.18 that $\sigma(T|_{\mathcal{R}(E_\lambda)}) = \{\lambda\}$. If $\lambda \neq 0$, then $0 \in \rho(T|_{\mathcal{R}(E_\lambda)})$, and so the compact operator $T|_{\mathcal{R}(E_\lambda)} \in \mathcal{B}_\infty[\mathcal{R}(E_\lambda)]$ is invertible, which implies that (cf. Proposition 1.Y)

$$\dim \mathcal{R}(E_\lambda) < \infty.$$

Set $m = \dim \mathcal{R}(E_\lambda)$. Note that $m \neq 0$ because $E_\lambda \neq O$ by the injectivity of Theorem 4.20. Since $\sigma(T|_{\mathcal{R}(E_\lambda)}) = \{\lambda\}$ we get by the Spectral Mapping Theorem (cf. Theorem 2.7) that $\sigma(\lambda I - T|_{\mathcal{R}(E_\lambda)}) = \{0\}$; that is, the operator $\lambda I - T|_{\mathcal{R}(E_\lambda)}$ is quasinilpotent on the m-dimensional space $\mathcal{R}(E_\lambda)$. This implies that $\lambda I - T|_{\mathcal{R}(E_\lambda)}$ in $\mathcal{B}[\mathcal{R}(E_\lambda)]$ is a nilpotent operator for which

$$(\lambda I - T|_{\mathcal{R}(E_\lambda)})^m = O.$$

Indeed, this is a purely finite-dimensional algebraic result that is obtained together with the well-known Cayley–Hamilton Theorem (see [48, Theorem 58.2]). Thus (since E_λ is a projection that commutes with T whose range is T-invariant), it follows that

$$(\lambda I - T)^m E_\lambda = O.$$

Moreover, since $\mathcal{X} \neq \{0\}$ we get $O \neq (\lambda I - T)^0 = I$, and so (since $E_\lambda \neq O$)

$$(\lambda I - T)^0 E_\lambda \neq O.$$

Thus there exists an integer $n \in [1, m]$ such that

$$(\lambda I - T)^n E_\lambda = O \quad \text{and} \quad (\lambda I - T)^{n-1} E_\lambda \neq O.$$

In other words,

$$\mathcal{R}(E_\lambda) \subseteq \mathcal{N}((\lambda I - T)^n) \quad \text{and} \quad \mathcal{R}(E_\lambda) \not\subseteq \mathcal{N}(\lambda I - T)^{n-1}.$$

(Recall that $AB = O$ if and only if $\mathcal{R}(B) \subseteq \mathcal{N}(A)$, for all operators A and B.) The remaining assertions are readily verified. In fact, $\{0\} \neq \mathcal{N}(\lambda I - T)$ since $\lambda \in \sigma(T)\backslash\{0\}$ is an eigenvalue of the compact operator T (by the Fredholm Alternative of Theorem 2.18), and $\mathcal{N}(\lambda I - T) \subseteq \mathcal{R}(E_\lambda)$ by Corollary 4.22. $\square$

Remark. Corollary 4.23 and Proposition 4.F ensure that, if $T \in \mathcal{B}_\infty[\mathcal{X}]$ (i.e., if T is compact) and $\lambda \in \sigma(T)\backslash\{0\}$, then λ is a pole of the resolvent function $R_T \colon \rho(T) \to \mathcal{G}[\mathcal{X}]$, and the integer n in the statement of the preceding result is the order of the pole λ. Moreover, it follows from Proposition 4.G that

$$\mathcal{R}(E_\lambda) = \mathcal{N}((\lambda I - T)^n).$$

4.5 Additional Propositions

Consider a set $\mathcal{S} \subseteq \mathcal{B}[\mathcal{H}]$ of operators on Hilbert space $\mathcal{H}$. The *commutant* $\mathcal{S}'$ of $\mathcal{S}$ is the set $\mathcal{S}' = \{T \in \mathcal{B}[\mathcal{H}] : TS = ST \text{ for every } S \in \mathcal{S}\}$ of all operators that commute with every operator in $\mathcal{S}$, which is a unital subalgebra of $\mathcal{B}[\mathcal{H}]$. The *double commutant* $\mathcal{S}''$ of $\mathcal{S}$ is the unital algebra $\mathcal{S}'' = (\mathcal{S}')'$. The DOUBLE COMMUTANT THEOREM says that *if* $\mathcal{A}$ *is a unital C*-subalgebra of* $\mathcal{B}[\mathcal{H}]$, *then* $\mathcal{A}^- = \mathcal{A}''$, where $\mathcal{A}^-$ stands for the weak closure of $\mathcal{A}$ in $\mathcal{B}[\mathcal{H}]$ (which coincides with the strong closure — see [27, Theorem IX.6.4]). A *von Neumann algebra* $\mathcal{A}$ is a C*-subalgebra of $\mathcal{B}[\mathcal{H}]$ such that $\mathcal{A} = \mathcal{A}''$, which is unital and is weakly (thus strongly) closed. Take a normal operator on a *separable* Hilbert space $\mathcal{H}$. Let $\mathcal{A}^*(T)$ be the *von Neumann algebra generated by* $T \in \mathcal{B}[\mathcal{H}]$, which is the intersection of all von Neumann algebras containing T, and coincides with the weak closure of $\mathcal{P}(T, T^*)$ (see the paragraph following Definition 3.12).

Proposition 4.A. *Consider the setup of* Theorem 4.8. *The range of* Φ_T *coincides with the von Neumann algebra generated by* T *(i.e.,* $\mathcal{R}(\Phi_T) = \mathcal{A}^*(T)$*).*

Proposition 4.B. *Take* $T \in \mathcal{B}[\mathcal{X}]$. *If* $\psi \in A(\sigma(T))$ *and* $\phi \in A(\sigma(\psi(T)))$, *then* $(\phi \circ \psi) \in A(\sigma(T))$ *and* $\phi(\psi(T)) = (\phi \circ \psi)(T)$. *That is,*

$$\phi(\psi(T)) = \tfrac{1}{2\pi i} \int_{\Gamma} \phi(\psi(\lambda)) \, R_T(\lambda) \, d\lambda,$$

where Γ *is any path about* $\sigma(T)$ *such that* $\sigma(T) \subset \mathrm{ins}\,\Gamma \subset (\mathrm{ins}\,\Gamma)^- \subset \Lambda$, *and the open set* $\Lambda \subseteq \mathbb{C}$ *is the domain of* $\phi \circ \psi$.

Proposition 4.C. *Let* $E_\Delta \in \mathcal{B}[\mathcal{X}]$ *be the Riesz idempotent associated with a spectral subset* Δ *of* $\sigma(T)$. *The point, residual, and continuous spectra of the restriction* $T|_{\mathcal{R}(E_\Delta)} \in \mathcal{B}[\mathcal{R}(E_\Delta)]$ *of* $T \in \mathcal{B}[\mathcal{X}]$ *to* $\mathcal{R}(E_\Delta)$ *are given by*

$$\sigma_P(T|_{\mathcal{R}(E_\Delta)}) = \Delta \cap \sigma_P(T), \;\; \sigma_R(T|_{\mathcal{R}(E_\Delta)}) = \Delta \cap \sigma_R(T), \;\; \sigma_C(T|_{\mathcal{R}(E_\Delta)}) = \Delta \cap \sigma_C(T).$$

Proposition 4.D. *If* λ_0 *is an isolated point of the spectrum* $\sigma(T)$ *of an operator* $T \in \mathcal{B}[\mathcal{X}]$, *then the range of the Riesz idempotent* $E_{\lambda_0} \in \mathcal{B}[\mathcal{X}]$ *associated with* λ_0 *is given by*

$$\mathcal{R}(E_{\lambda_0}) = \big\{ x \in \mathcal{X} : \|(\lambda_0 I - T)^n x\|^{\frac{1}{n}} \to 0 \big\}.$$

Proposition 4.E. *If* λ_0 *is an isolated point of the spectrum* $\sigma(T)$ *of an operator* $T \in \mathcal{B}[\mathcal{X}]$, *then (with convergence in* $\mathcal{B}[\mathcal{X}]$*)*

$$R_T(\lambda) = (\lambda I - T)^{-1} = \sum_{k=-\infty}^{\infty} (\lambda - \lambda_0)^k T_k$$
$$= \sum_{k=1}^{\infty} (\lambda - \lambda_0)^{-k} T_{-k} + T_0 + \sum_{k=1}^{\infty} (\lambda - \lambda_0)^k T_k$$

for every λ *in the punctured disk* $B_{\delta_0}(\lambda_0) \backslash \{0\} = \{\lambda \in \mathbb{C} : 0 < |\lambda - \lambda_0| < \delta_0\} \subseteq \rho(T)$, *where* $\delta_0 = d(\lambda_0, \sigma(T) \backslash \{\lambda_0\})$ *is the distance from* λ_0 *to the rest of the spectrum, and* $T_k \in \mathcal{B}[\mathcal{X}]$ *is such that*

$$T_k = \tfrac{1}{2\pi i} \int_{\Gamma_{\lambda_0}} (\lambda - \lambda_0)^{-(k+1)} R_T(\lambda)\, d\lambda$$

for every $k \in \mathbb{Z}$, where Γ_{λ_0} is any positively oriented circle centered at λ_0 with radius less than δ_0.

An isolated point of $\sigma(T)$ is a *singularity* (or an *isolated singularity*) of the resolvent function $R_T \colon \rho(T) \to \mathcal{G}[\mathcal{X}]$. The expansion of R_T in Proposition 4.E is the *Laurent expansion* of R_T about an isolated point of the spectrum. Note that, for every positive integer k (i.e., for every $k \in \mathbb{N}$),

$$T_{-k} = E_{\lambda_0}(T - \lambda_0 I)^{k-1} = (T - \lambda_0 I)^{k-1} E_{\lambda_0},$$

and so $T_{-1} = E_{\lambda_0}$, the Riesz idempotent associated with the isolated point λ_0. The notion of poles associated with isolated singularities of analytic functions is extended as follows. An isolated point of $\sigma(T)$ is a *pole of order n* of R_T, for sone $n \in \mathbb{N}$, if $T_{-n} \neq O$ and $T_k = O$ for every $k < n$ (i.e., n is the largest positive integer such that $T_{-|n|} \neq O$). Otherwise, if the number of nonzero coefficients T_k with negative indices is infinite, then the isolated point λ_0 of $\sigma(T)$ is said to be an *essential singularity* of R_T.

Proposition 4.F. *Let λ_0 be an isolated point of the spectrum $\sigma(T)$ of $T \in \mathcal{B}[\mathcal{X}]$, and let $E_{\lambda_0} \in \mathcal{B}[\mathcal{X}]$ be the Riesz idempotent associated with λ_0.*

(a) *The isolated point λ_0 is a pole of R_T of order n if and only if*

$$(\lambda_0 I - T)^n E_{\lambda_0} = O \quad and \quad (\lambda_0 I - T)^{n-1} E_{\lambda_0} \neq O.$$

Therefore, if λ_0 is a pole of R_T of order n, then

(b) $$\{0\} \neq \mathcal{R}((\lambda_0 I - T)^{n-1} E_{\lambda_0}) \subseteq \mathcal{N}(\lambda_0 I - T),$$

and so λ_0 is an eigenvalue of T (i.e., $\lambda_0 \in \sigma_P(T)$). Actually (by item (a) and Corollary 4.22), λ_0 is a pole of R_T of order 1 if and only if

(c) $$\{0\} \neq \mathcal{R}(E_{\lambda_0}) = \mathcal{N}(\lambda_0 I - T).$$

Proposition 4.G. *Take $T \in \mathcal{B}[\mathcal{X}]$. If λ_0 is a pole of R_T of order n, then*

$$\mathcal{R}(E_{\lambda_0}) = \mathcal{N}((\lambda_0 I - T)^n) \quad and \quad \mathcal{R}(E_\Delta) = \mathcal{R}((\lambda_0 I - T)^n),$$

where E_{λ_0} denotes the Riesz idempotent associated with λ_0 and E_Δ denotes the Riesz idempotent associated with the spectral set $\Delta = \sigma(T) \backslash \{\lambda_0\}$.

Proposition 4.H. *Let E_Δ be the Riesz idempotent associated with a spectral subset Δ of $\sigma(T)$. If $\dim \mathcal{R}(E_\Delta) < \infty$, then Δ is a finite set of poles.*

Proposition 4.I. *Take $T \in \mathcal{B}[\mathcal{X}]$. Let Δ be a spectral subset of $\sigma(T)$, and let $E_\Delta \in \mathcal{B}[\mathcal{X}]$ be the Riesz idempotent associated with Δ. If $\psi \in \mathcal{A}(\sigma(T))$, then $\psi \in \mathcal{A}(\sigma(T|_{\mathcal{R}(E_\Delta)}))$ and $\psi(T)|_{\mathcal{R}(E_\Delta)} = \psi(T|_{\mathcal{R}(E_\Delta)})$, so that*

$$\frac{1}{2\pi i} \int_{\Gamma_\Delta} \psi(\lambda) R_T(\lambda)\, d\lambda \Big|_{\mathcal{R}(E_\Delta)} = \frac{1}{2\pi i} \int_{\Gamma_\Delta} \psi(\lambda) R_{T|_{\mathcal{R}(E_\Delta)}}(\lambda)\, d\lambda.$$

For a normal operator on a complex Hilbert space, the functional calculus of Sections 4.2 and 4.3 (cf. Theorems 4.2 and 4.16) coincides for every analytic function on a neighborhood of the spectrum.

Proposition 4.J. *If $T = \int \lambda\, dE_\lambda \in \mathcal{B}[\mathcal{H}]$ is the spectral decomposition of a normal operator T on a Hilbert space $\mathcal{H}$ and if $\psi \in A(\sigma(T))$, then*

$$\psi(T) = \int \psi(\lambda)\, dE_\lambda = \tfrac{1}{2\pi i} \int_\Gamma \psi(\lambda)\, R_T(\lambda)\, d\lambda,$$

with the first integral as in Lemma 3.7 *and the second as in* Definition 4.14.

Consider Definition 3.6. The notion of *spectral measure* $E \colon \mathcal{A}_\Omega \to \mathcal{B}[\mathcal{X}]$ (carrying the same properties of Definition 3.6) can be extended to a (complex) Banach space $\mathcal{X}$ where the projections $E(\Lambda)$ for $\Lambda \in \mathcal{A}_\Omega$ are not orthogonal but are bounded. If an operator $T \in \mathcal{B}[\mathcal{X}]$ is such that $E(\Lambda)T = TE(\Lambda)$ and $\sigma(T|_{\mathcal{R}(E(\Lambda))}) \subseteq \Lambda^-$ (see Lemma 3.16), then T is called a *spectral operator* (cf. [41, Definition XV.2.5]). Take $\psi \in B(\Omega)$, a bounded $\mathcal{A}_\Omega$-measurable complex-valued function on Ω. An integral $\int \psi(\lambda)\, dE_\lambda$ can be defined in this Banach space setting as the uniform limit of integrals $\int \phi_n(\lambda)\, dE_\lambda = \sum_{i=1}^n \alpha_i E(\Lambda_i)$ of measurable *simple functions* $\phi_n = \sum_{i=1}^n \alpha_i \chi_{\Lambda_i}$ (i.e., of finite linear combinations of characteristic functions χ_{Λ_i} of measurable sets $\Lambda_i \in \mathcal{A}_\Omega$) — see, e.g., [40, pp. 891–893]. If a spectral operator $T \in \mathcal{B}[\mathcal{X}]$ is such that $T = \int \lambda\, dE_\lambda$, then it is said to be of *scalar type* (cf. [41, Definition XV.4.1]). Proposition 4.J holds for Banach space *spectral operators of scalar type*. In this case, the preceding integral $\int \lambda\, dE_\lambda = \int \varphi(\lambda)\, dE_\lambda$ (of the function $\varphi(\lambda) = \lambda \in \mathbb{C}$ for every $\lambda \in \Omega$, defined as the limit of a sequence $\sum_{i=1}^n \beta_i E(\Lambda_i)$ of integrals of simple functions) coincides with the integral of Theorem 3.15 if $\mathcal{X}$ is a Hilbert space, and if the spectral measure $E \colon \mathcal{A}_\Omega \to \mathcal{B}[\mathcal{X}]$, with $\Omega = \sigma(T)$, takes on orthogonal projections only.

Proposition 4.K. *If $T = \int \lambda\, dE_\lambda \in \mathcal{B}[\mathcal{H}]$ is the spectral decomposition of a normal operator T on a Hilbert space $\mathcal{H}$, then the orthogonal projection $E(\Lambda)$ coincides with the Riesz idempotent E_Λ for every clopen Λ in $\mathcal{A}_{\sigma(T)}$:*

$$E(\Lambda) = \int_\Lambda dE_\lambda = \tfrac{1}{2\pi i} \int_{\Gamma_\Lambda} R_T(\lambda)\, d\lambda = E_\Lambda.$$

In particular, let $T = \sum_k \lambda_k E_k$ be the spectral decomposition of a *compact normal* operator T on a Hilbert space, with $\{\lambda_k\} = \sigma_P(T)$ and $\{E_k\}$ being a countable resolution of the identity where each orthogonal projection E_k is such that $\mathcal{R}(E_k) = \mathcal{N}(\lambda_k I - T)$ (cf. Theorem 3.3). Take any nonzero eigenvalue λ_k in $\sigma(T)\setminus\{0\} = \sigma_P(T)\setminus\{0\} = \{\lambda_k\}\setminus\{0\}$ so that $\dim \mathcal{N}(\lambda_k I - T) < \infty$ (cf. Theorems 1.19 and 2.18 and Corollary 2.20). Each orthogonal projection E_k coincides with the Riesz idempotent E_{λ_k} associated with each isolated point

$0 \neq \lambda_k$ of $\sigma(T)$ according to Proposition 4.K. Therefore, by Corollary 4.23 and Proposition 4.G, it follows that $\mathcal{N}(\lambda_k I - T) = \mathcal{N}((\lambda_k I - T)^{n_k}) = \mathcal{R}(E_{\lambda_k})$, where the integer n_k is the order of the pole λ_k and the integer $\dim \mathcal{R}(E_{\lambda_k}) = \dim \mathcal{N}(\lambda_k I - T) < \infty$ is the finite multiplicity of the eigenvalue λ_k. Note that $\dim \mathcal{R}(E_{\lambda_k})$ is sometimes called the *algebraic multiplicity* of the isolated point λ_k of the spectrum, while $\dim \mathcal{N}(\lambda_k I - T)$ — *the multiplicity* of the eigenvalue λ_k — is sometimes referred to as the *geometric multiplicity* of λ_k. In the preceding case (i.e., if T is compact and normal), these multiplicities are finite and coincide.

Proposition 4.L. *Every isolated point of the spectrum of a hyponormal operator is an eigenvalue. In fact, if λ_0 is an isolated point of $\sigma(T)$, then*

$$\{0\} \neq \mathcal{R}(E_{\lambda_0}) \subseteq \mathcal{N}(\lambda_0 I - T)$$

for every hyponormal operator $T \in \mathcal{B}[\mathcal{H}]$, where E_{λ_0} is the Riesz idempotent associated with the isolated point λ_0 of the spectrum of T.

Notes: For Proposition 4.A see, for instance, [27, Theorem IX.8.10]. Proposition 4.B is the composition counterpart of the product-preserving result of Lemma 4.15 (see, e.g., [56, Theorem 5.3.2] or [39, Theorem VII.3.12]). For Proposition 4.C see [39, Exercise VII.5.18] or [87, 1st edn. Theorem 5.7-B]. Proposition 4.D follows from Theorem 4.20, since, by Theorem 2.7, $\lambda_0 I - T|_{\mathcal{R}(E_{\lambda_0})}$ is quasinilpotent so that $\lim_n \|(\lambda I - T)^n x\|^{\frac{1}{n}} = 0$ whenever $x \in \mathcal{R}(E_\lambda)$. For the converse see [87, 1st edn. Lemma 5.8-C]. The Laurent expansion in Proposition 4.E and also Proposition 4.F are standard results (see [27, Lemma VII.6.11, Proposition 6.12, and Corollary 6.13]). Proposition 4.G refines the results of Proposition 4.F — see, e.g., [39, Theorem VII.3.24]. For Proposition 4.H see [39, Exercise VII.5.34]. Proposition 4.I is a consequence of Theorem 4.20 (see, e.g., [39, Theorem VII.3.20]). Proposition 4.J also holds for Banach space spectral operators of the scalar type (cf. [41, Theorem XV.5.1]), and Proposition 4.K is a particular case of it for the characteristic function in a neighborhood of a clopen set in the σ-algebra of Borel subsets of $\sigma(T)$. Proposition 4.L is the extension of Proposition 3.G from normal to hyponormal operators, which is obtained by the Riesz Decomposition Theorem (Corollary 4.21) — see, for instance, [66, Problem 6.28].

Suggested Reading

Arveson [6]
Brown and Pearcy [21]
Conway [27, 29]
Dowson [35]
Dunford and Schwartz [39, 40, 41]

Hille and Phillips [56]
Sz.-Nagy and Foiaş [86]
Radjavi and Rosenthal [76]
Riesz and Sz.-Nagy [77]
Taylor and Lay [87]

5

Fredholm Theory

The central theme of this chapter investigates compact perturbations. We shall
be particularly concerned with properties of the spectrum of an operator that
are invariant under compact perturbations; that is, properties of the spectrum
of T that are also possessed by the spectrum of $T + K$ for every compact
operator K. As in Sections 1.8 and 2.6, we assume throughout this chapter
that all operators lie in $\mathcal{B}[\mathcal{H}]$, where $\mathcal{H}$ stands for a nonzero complex Hilbert
space, although part of the theory developed here applies equally for operators
on Banach spaces.

5.1 Fredholm Operators and Fredholm Index

Let $\mathcal{B}_\infty[\mathcal{H}]$ be the (two-sided) ideal of all compact operators from $\mathcal{B}[\mathcal{H}]$. An op-
erator $T \in \mathcal{B}[\mathcal{H}]$ is *left semi-Fredholm* if there exist $A \in \mathcal{B}[\mathcal{H}]$ and $K \in \mathcal{B}_\infty[\mathcal{H}]$
such that $AT = I + K$, and *right semi-Fredholm* if there exist $A \in \mathcal{B}[\mathcal{H}]$ and
$K \in \mathcal{B}_\infty[\mathcal{H}]$ such that $TA = I + K$. We say that $T \in \mathcal{B}[\mathcal{H}]$ is *semi-Fredholm*
if it is either left or right semi-Fredholm, and *Fredholm* if it is both left and
right semi-Fredholm. Let $\mathcal{F}_\ell$ and $\mathcal{F}_r$ be the classes of all left semi-Fredholm
operators and of all right semi-Fredholm operators:

$$\mathcal{F}_\ell = \big\{T \in \mathcal{B}[\mathcal{H}] \colon AT = I + K \text{ for some } A \in \mathcal{B}[\mathcal{H}] \text{ and some } K \in \mathcal{B}_\infty[\mathcal{H}]\big\},$$

$$\mathcal{F}_r = \big\{T \in \mathcal{B}[\mathcal{H}] \colon TA = I + K \text{ for some } A \in \mathcal{B}[\mathcal{H}] \text{ and some } K \in \mathcal{B}_\infty[\mathcal{H}]\big\}.$$

The classes of all semi-Fredholm and Fredholm operators from $\mathcal{B}[\mathcal{H}]$ will be
denoted by $\mathcal{SF}$ and $\mathcal{F}$, respectively;

$$\mathcal{SF} = \mathcal{F}_\ell \cup \mathcal{F}_r \quad \text{and} \quad \mathcal{F} = \mathcal{F}_\ell \cap \mathcal{F}_r.$$

According to Proposition 1.W, $K \in \mathcal{B}_\infty[\mathcal{H}]$ if and only if $K^* \in \mathcal{B}_\infty[\mathcal{H}]$. Thus

$$T \in \mathcal{F}_\ell \quad \text{if and only if} \quad T^* \in \mathcal{F}_r.$$

Therefore,

$$T \in \mathcal{SF} \quad \text{if and only if} \quad T^* \in \mathcal{SF},$$

$$T \in \mathcal{F} \quad \text{if and only if} \quad T^* \in \mathcal{F}.$$

The following properties on kernel and range of a Hilbert space operator and its adjoint (Lemmas 1.4 and 1.5) will often be used throughout this chapter:

$$\mathcal{N}(T^*) = \mathcal{H} \ominus \mathcal{R}(T)^- = \mathcal{R}(T)^\perp,$$

$$\mathcal{R}(T^*) \ \text{is closed if and only if} \ \mathcal{R}(T) \ \text{is closed}.$$

Theorem 5.1. (a) *An operator $T \in \mathcal{B}[\mathcal{H}]$ is left semi-Fredholm if and only if $\mathcal{R}(T)$ is closed and $\mathcal{N}(T)$ is finite dimensional.*

(b) *Hence,*

$$\mathcal{F}_\ell = \big\{T \in \mathcal{B}[\mathcal{H}] \colon \mathcal{R}(T) \ \text{is closed} \ \text{and} \ \dim \mathcal{N}(T) < \infty \big\},$$

$$\mathcal{F}_r = \big\{T \in \mathcal{B}[\mathcal{H}] \colon \mathcal{R}(T) \ \text{is closed} \ \text{and} \ \dim \mathcal{N}(T^*) < \infty \big\}.$$

Proof. Let A, T, and K be operators on $\mathcal{H}$, where K is compact.

(a_1) If T is left semi-Fredholm, then there are operators A and K such that $AT = I + K$, and so $\mathcal{N}(T) \subseteq \mathcal{N}(AT) = \mathcal{N}(I + K)$ and $\mathcal{R}(AT) = \mathcal{R}(I + K)$. The Fredholm Alternative (Corollary 1.20) says that $\dim \mathcal{N}(I + K) < \infty$ and $\mathcal{R}(I + K)$ is closed. Therefore,

(i) $$\dim \mathcal{N}(T) < \infty,$$

(ii) $$\dim T(\mathcal{N}(AT)) < \infty,$$

(iii) $$\mathcal{R}(AT) \ \text{is closed}.$$

This implies that

(iv) $$T(\mathcal{N}(AT)^\perp) \ \text{is closed}.$$

Indeed, the restriction $(AT)|_{\mathcal{N}(AT)^\perp} \colon \mathcal{N}(AT)^\perp \to \mathcal{H}$ is bounded below by (iii) since $(AT)|_{\mathcal{N}(AT)^\perp}$ is injective with range $\mathcal{R}(AT)$ (Theorem 1.2). Thus there exists an $\alpha > 0$ such that $\alpha \|v\| \leq \|ATv\| \leq \|A\| \, \|Tv\|$ for every $v \in \mathcal{N}(AT)^\perp$. Then $T|_{\mathcal{N}(AT)^\perp} \colon \mathcal{N}(AT)^\perp \to \mathcal{H}$ is bounded below, and so $T|_{\mathcal{N}(AT)^\perp}$ has a closed range (Theorem 1.2 again), proving (iv). But (ii) and (iv) imply that

(v) $$\mathcal{R}(T) \ \text{is closed}.$$

In fact, since $\mathcal{H} = \mathcal{N}(AT) + \mathcal{N}(AT)^\perp$ by the Projection Theorem, it follows that $\mathcal{R}(T) = T(\mathcal{H}) = T(\mathcal{N}(AT) + \mathcal{N}(AT)^\perp) = T(\mathcal{N}(AT)) + T(\mathcal{N}(AT)^\perp)$. Thus assertions (ii) and (iv) ensure that $\mathcal{R}(T)$ is closed (since the sum of a closed linear manifold and a finite-dimensional linear manifold is closed —

see Proposition 1.C). This concludes the proof of (v) which, together with (i), concludes half of the claimed result.

(a_2) Conversely, suppose $\dim \mathcal{N}(T) < \infty$ and $\mathcal{R}(T)$ is closed. Since $\mathcal{R}(T)$ is closed, it follows that $T|_{\mathcal{N}(T)^{\perp}} : \mathcal{N}(T)^{\perp} \to \mathcal{H}$ (which is injective because $\mathcal{N}(T|_{\mathcal{N}(T)^{\perp}}) = \{0\}$) has a closed range $\mathcal{R}(T|_{\mathcal{N}(T)^{\perp}}) = \mathcal{R}(T)$. Hence it has a bounded inverse on its range (Theorem 1.2). Let $E \in \mathcal{B}[\mathcal{H}]$ be the orthogonal projection onto $\mathcal{R}(E) = \mathcal{R}(T|_{\mathcal{N}(T)^{\perp}}) = \mathcal{R}(T)$, and define $A \in \mathcal{B}[\mathcal{H}]$ as follows:

$$A = (T|_{\mathcal{N}(T)^{\perp}})^{-1} E.$$

If $u \in \mathcal{N}(T)$, then $ATu = 0$ trivially. On the other hand, if $v \in \mathcal{N}(T)^{\perp}$, then $ATv = (T|_{\mathcal{N}(T)^{\perp}})^{-1} ET|_{\mathcal{N}(T)^{\perp}} v = (T|_{\mathcal{N}(T)^{\perp}})^{-1} T|_{\mathcal{N}(T)^{\perp}} v = v$. Thus, for every $x = u + v$ in $\mathcal{H} = \mathcal{N}(T) + \mathcal{N}(T)^{\perp}$, we get $ATx = ATu + ATv = v = E'x$, where $E' \in \mathcal{B}[\mathcal{H}]$ is the orthogonal projection onto $\mathcal{N}(T)^{\perp}$. Therefore,

$$AT = I + K,$$

where $-K = I - E' \in \mathcal{B}[\mathcal{H}]$ is the complementary orthogonal projection onto the finite-dimensional space $\mathcal{N}(T)$. Hence K is a finite-rank operator, and so compact (Proposition 1.X). Thus T is left semi-Fredholm.

(b) Recalling that $T \in \mathcal{F}_{\ell}$ if and only if $T^* \in \mathcal{F}_r$, and $\mathcal{R}(T^*)$ is closed if and only if $\mathcal{R}(T)$ is closed (Lemma 1.5), item (b) follows from item (a). $\qquad \square$

Corollary 5.2. *Take an operator $T \in \mathcal{B}[\mathcal{H}]$.*

(a) *T is semi-Fredholm if and only if $\mathcal{R}(T)$ is closed and $\mathcal{N}(T)$ or $\mathcal{N}(T^*)$ is finite dimensional.*

(b) *T is Fredholm if and only if $\mathcal{R}(T)$ is closed and both $\mathcal{N}(T)$ and $\mathcal{N}(T^*)$ are finite dimensional.*

In other words,

$$\mathcal{SF} = \big\{ T \in \mathcal{B}[\mathcal{H}] \colon \mathcal{R}(T) \text{ is closed, } \dim \mathcal{N}(T) < \infty \ \text{ or } \ \dim \mathcal{N}(T^*) < \infty \big\},$$

$$\mathcal{F} = \big\{ T \in \mathcal{B}[\mathcal{H}] \colon \mathcal{R}(T) \text{ is closed, } \dim \mathcal{N}(T) < \infty \ \text{ and } \ \dim \mathcal{N}(T^*) < \infty \big\}.$$

Therefore, the complement of $\mathcal{F}$,

$$\mathcal{B}[\mathcal{H}] \backslash \mathcal{F} = \mathcal{B}[\mathcal{H}] \backslash (\mathcal{F}_{\ell} \cap \mathcal{F}_r) = (\mathcal{B}[\mathcal{H}] \backslash \mathcal{F}_{\ell}) \cup (\mathcal{B}[\mathcal{H}] \backslash \mathcal{F}_r),$$

is the union of

$$\mathcal{B}[\mathcal{H}] \backslash \mathcal{F}_{\ell} = \big\{ T \in \mathcal{B}[\mathcal{H}] \colon \mathcal{R}(T) \text{ is not closed } \ \text{or} \ \dim \mathcal{N}(T) = \infty \big\},$$

$$\mathcal{B}[\mathcal{H}] \backslash \mathcal{F}_r = \big\{ T \in \mathcal{B}[\mathcal{H}] \colon \mathcal{R}(T) \text{ is not closed } \ \text{or} \ \dim \mathcal{N}(T^*) = \infty \big\}.$$

Proof. Recall that $\mathcal{SF} = \mathcal{F}_{\ell} \cup \mathcal{F}_r$ and $\mathcal{F} = \mathcal{F}_{\ell} \cap \mathcal{F}_r$. Apply Theorem 5.1. $\quad \square$

Let $\mathbb{Z}$ be the set of all integers and set $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty\} \cup \{+\infty\}$, the extended integers. Take any T in $\mathcal{SF}$ so that $\mathcal{N}(T)$ or $\mathcal{N}(T^*)$ is finite dimensional. The *Fredholm index* $\mathrm{ind}\,(T)$ of T in $\mathcal{SF}$ is defined in $\overline{\mathbb{Z}}$ by

$$\mathrm{ind}\,(T) = \dim \mathcal{N}(T) - \dim \mathcal{N}(T^*).$$

It is usual to write $\alpha(T) = \dim \mathcal{N}(T)$ and $\beta(T) = \dim \mathcal{N}(T^*)$, and hence $\mathrm{ind}\,(T) = \alpha(T) - \beta(T)$. Since T^* and T lie in $\mathcal{SF}$ together, we get

$$\mathrm{ind}\,(T^*) = -\mathrm{ind}\,(T).$$

Note: $\beta(T) = \alpha(T^*) = \dim \mathcal{R}(T)^{\perp} = \dim(\mathcal{H} \ominus \mathcal{R}(T)^-) = \dim(\mathcal{H}/\mathcal{R}(T)^-)$, where the last identity is explained in the forthcoming Remark 5.3(a).

Recall from Theorem 5.1 that $T \in \mathcal{F}_\ell$ implies $\dim \mathcal{N}(T) < \infty$ and, dually, $T \in \mathcal{F}_r$ implies $\dim \mathcal{N}(T^*) < \infty$. Therefore,

$$T \in \mathcal{F}_\ell \quad \text{implies} \quad \mathrm{ind}\,(T) \neq +\infty,$$

$$T \in \mathcal{F}_r \quad \text{implies} \quad \mathrm{ind}\,(T) \neq -\infty.$$

In other words, if $T \in \mathcal{SF} = \mathcal{F}_\ell \cup \mathcal{F}_r$, then

$$\mathrm{ind}\,(T) = +\infty \quad \text{implies} \quad T \in \mathcal{F}_r \backslash \mathcal{F}_\ell = \mathcal{F}_r \backslash \mathcal{F},$$

$$\mathrm{ind}\,(T) = -\infty \quad \text{implies} \quad T \in \mathcal{F}_\ell \backslash \mathcal{F}_r = \mathcal{F}_\ell \backslash \mathcal{F}.$$

Remark 5.3. (a) BANACH SPACE. If T is an operator on a Hilbert space $\mathcal{H}$, then $\mathcal{N}(T^*) = \mathcal{R}(T)^{\perp} = \mathcal{H} \ominus \mathcal{R}(T)^-$ by Lemma 1.4. Thus Theorem 5.1 and Corollary 5.2 can be restated with $\mathcal{N}(T^*)$ replaced by $\mathcal{H} \ominus \mathcal{R}(T)^-$. However, consider the quotient space $\mathcal{H}/\mathcal{R}(T)^-$ of $\mathcal{H}$ modulo $\mathcal{R}(T)^-$, consisting of all cosets $x + \mathcal{R}(T)^-$ for each $x \in \mathcal{H}$. The natural mapping of $\mathcal{H}/\mathcal{R}(T)^-$ onto $\mathcal{H} \ominus \mathcal{R}(T)^-$ (viz., $x + \mathcal{R}(T)^- \mapsto Ex$ where E is the orthogonal projection on $\mathcal{R}(T)^{\perp}$) is an isomorphism. Then $\dim(\mathcal{H}/\mathcal{R}(T)^-) = \dim(\mathcal{H} \ominus \mathcal{R}(T)^-)$. Therefore we get the following restatement of Corollary 5.2.

> $T \in \mathcal{B}[\mathcal{H}]$ *is semi-Fredholm if and only if* $\mathcal{R}(T)$ *is closed and* $\mathcal{N}(T)$ *or* $\mathcal{H}/\mathcal{R}(T)$ *is finite dimensional. It is Fredholm if and only if* $\mathcal{R}(T)$ *is closed and both* $\mathcal{N}(T)$ *and* $\mathcal{H}/\mathcal{R}(T)$ *are finite dimensional.*

This is how the theory advances in a Banach space. Many properties, among those that will be developed in this chapter, work smoothly in any Banach space. Some of them, in order to be properly translated into a Banach space setting, will behave well if the Banach space is complemented. A Banach space is *complemented* if every subspace of it has a complementary subspace (i.e., if there exists a subspace $\mathcal{N}$ of $\mathcal{X}$ such that $\mathcal{M} + \mathcal{N} = \mathcal{X}$ and $\mathcal{M} \cap \mathcal{N} = \{0\}$ for every subspace $\mathcal{M}$ of $\mathcal{X}$). However, *if a Banach space is complemented, then it is isomorphic to a Hilbert space* [72] (i.e., topologically isomorphic, after the

Inverse Mapping Theorem). For instance, Proposition 1.D (on complementary subspaces and continuous projections) is a significant exemplar of a result that acquires its full strength only on complemented Banach spaces (which are identified with Hilbert spaces). Only Hilbert spaces (up to an isomorphism) are complemented. We shall stick to Hilbert spaces.

(b) FINITE RANK. It was actually proved in Theorem 5.1 that

> *T is left semi-Fredholm if and only if there exists an operator $A \in \mathcal{B}[\mathcal{H}]$ and a* finite-rank *operator $K \in \mathcal{B}[\mathcal{H}]$ such that $AT = I + K$.*

Since T is right semi-Fredholm if and only if T^* is left semi-Fredholm, which means that there exists a finite-rank operator K such that $AT^* = I + K$ or, equivalently, such that $TA^* = I + K^*$, and since K is a finite-rank operator if and only if K^* is (see, e.g., [66, Problem 5.40]), it follows that

> *T is right semi-Fredholm if and only if there exists an operator $A \in \mathcal{B}[\mathcal{H}]$ and a* finite-rank *operator $K \in \mathcal{B}[\mathcal{H}]$ such that $TA = I + K$.*

So the definitions of left semi-Fredholm, right semi-Fredholm, semi-Fredholm, and Fredholm operators are equivalently stated if "compact" is replaced with "finite-rank". For Fredholm operators we can even have *the same A when stating that $I - AT$ and $I - TA$ are of finite rank* (cf. Proposition 5.C).

(c) FINITE DIMENSIONAL. Take any operator $T \in \mathcal{B}[\mathcal{H}]$. Recall from linear algebra that $\dim \mathcal{H} = \dim \mathcal{N}(T) + \dim \mathcal{R}(T)$ (see, e.g., [66, Problem 2.17]. Since $\mathcal{H} = \mathcal{R}(T)^- \oplus \mathcal{R}(T)^\perp$ by the Projection Theorem, it follows that $\dim \mathcal{H} = \dim \mathcal{R}(T) + \dim \mathcal{R}(T)^\perp$. Thus, if $\mathcal{H}$ is finite dimensional (so that $\mathcal{N}(T)$ and $\mathcal{N}(T^*)$ are finite dimensional), then $\mathrm{ind}\,(T) = 0$. Indeed, if $\dim \mathcal{H} < \infty$, then

$$
\begin{aligned}
\dim \mathcal{N}(T) - \dim \mathcal{N}(T^*) &= \dim \mathcal{N}(T) - \dim \mathcal{R}(T)^\perp \\
&= \dim \mathcal{N}(T) + \dim \mathcal{R}(T) - \dim \mathcal{H} \\
&= \dim \mathcal{H} - \dim \mathcal{H} = 0.
\end{aligned}
$$

Since linear manifolds of finite-dimensional spaces are always closed, it follows by Corollary 5.2 that *on a finite-dimensional space every operator is Fredholm with a null index*. That is,

$$
\dim \mathcal{H} < \infty \quad \Longrightarrow \quad \big\{ T \in \mathcal{F}\colon \mathrm{ind}\,(T) = 0 \big\} = \mathcal{B}[\mathcal{H}].
$$

(d) FREDHOLM ALTERNATIVE. If $K \in \mathcal{B}_\infty[\mathcal{H}]$ and $\lambda \neq 0$, then $\mathcal{R}(\lambda I - K)$ is closed and $\dim \mathcal{N}(\lambda I - K) = \dim \mathcal{N}(\overline{\lambda} I - K^*) < \infty$. This is the Fredholm Alternative for compact operators of Corollary 1.20, which can be restated in terms of Fredholm indices, according to Corollary 5.2, as follows.

> *If $K \in \mathcal{B}_\infty[\mathcal{H}]$ and $\lambda \neq 0$, then $\lambda I - K$ is Fredholm with $\mathrm{ind}\,(\lambda I - K) = 0$.*

But $\mathrm{ind}\,(\lambda I - K) = 0$ means $\dim \mathcal{N}(\lambda I - K) = \dim \mathcal{R}(\lambda I - K)^\perp$ (recall that $\dim \mathcal{N}(\overline{\lambda} I - K^*) = \dim \mathcal{R}(\lambda I - K)^\perp$ by Lemma 1.5), and this implies that

$\mathcal{N}(\lambda I - K) = \{0\}$ if and only if $\mathcal{R}(\lambda I - K)^- = \mathcal{H}$ (equivalently, if and only if $\mathcal{R}(\lambda I - K)^\perp = \{0\}$). Since $\mathcal{R}(\lambda I - K)$ is closed if $\lambda I - K$ is Fredholm, we get another form of the Fredholm Alternative (cf. Theorems 1.18 and 2.18).

> If $K \in \mathcal{B}_\infty[\mathcal{H}]$ *and* $\lambda \neq 0$, *then* $\mathcal{N}(\lambda I - K) = \{0\} \iff \mathcal{R}(\lambda I - K) = \mathcal{H}$,

which means that $\lambda I - K$ *is injective if and only if it is surjective.*

(e) FREDHOLM INDEX. Corollary 5.2 and some of its straightforward consequences can also be naturally rephrased in terms of Fredholm indices. Note that $\dim \mathcal{N}(T)$ and $\dim \mathcal{N}(T^*)$ are both finite if and only if $\operatorname{ind}(T)$ is finite (reason: $\operatorname{ind}(T)$ was defined for semi-Fredholm operators only, so that if one of $\dim \mathcal{N}(T)$ or $\dim \mathcal{N}(T^*)$ is infinite, then the other must be finite). Thus, T *is Fredholm if and only if it is semi-Fredholm with a finite index*, and so

$$T \in \mathcal{SF} \quad \Longrightarrow \quad \{T \in \mathcal{F} \iff |\operatorname{ind}(T)| < \infty\}.$$

Since $\mathcal{F}_\ell \backslash \mathcal{F}_r = \mathcal{F}_\ell \backslash \mathcal{F}$ and $\mathcal{F}_r \backslash \mathcal{F}_\ell = \mathcal{F}_r \backslash \mathcal{F}$, it follows that

$$T \in \mathcal{SF} \quad \Longrightarrow \quad \{T \in \mathcal{F}_r \backslash \mathcal{F}_\ell \iff \operatorname{ind}(T) = +\infty\},$$

$$T \in \mathcal{SF} \quad \Longrightarrow \quad \{T \in \mathcal{F}_\ell \backslash \mathcal{F}_r \iff \operatorname{ind}(T) = -\infty\}.$$

(f) PRODUCT. Still from Corollary 5.2, a nonzero scalar operator is Fredholm with a null index. This is readily generalized:

> If $T \in \mathcal{F}$, *then* $\gamma T \in \mathcal{F}$ *and* $\operatorname{ind}(\gamma T) = \operatorname{ind}(T)$ *for every* $\gamma \in \mathbb{C} \backslash \{0\}$.

A further generalization leads to the most important property of the index, namely its *logarithmic additivity*: $\operatorname{ind}(ST) = \operatorname{ind}(S) + \operatorname{ind}(T)$ whenever such an addition makes sense.

Theorem 5.4. *Take* $S, T \in \mathcal{B}[\mathcal{H}]$.

(a) *If* $S, T \in \mathcal{F}_\ell$, *then* $ST \in \mathcal{F}_\ell$. *If* $S, T \in \mathcal{F}_r$, *then* $ST \in \mathcal{F}_r$.
  (*Therefore, if* $S, T \in \mathcal{F}$, *then* $ST \in \mathcal{F}$.)

(b) *If* $S, T \in \mathcal{F}_\ell$ *or* $S, T \in \mathcal{F}_r$ (*in particular, if* $S, T \in \mathcal{F}$), *then*

$$\operatorname{ind}(ST) = \operatorname{ind}(S) + \operatorname{ind}(T).$$

Proof. (a) If S and T are left semi-Fredholm, then there are A, B, K, L in $\mathcal{B}[\mathcal{H}]$, being K, L compact, such that $BS = I + L$ and $AT = I + K$. Then

$$(AB)(ST) = A(I + L)T = AT + ALT = I + (K + ALT).$$

But $K + ALT$ is compact (because $\mathcal{B}_\infty[\mathcal{H}]$ is an ideal of $\mathcal{B}[\mathcal{H}]$). Thus ST is left semi-Fredholm. Summing up: $S, T \in \mathcal{F}_\ell$ implies $ST \in \mathcal{F}_\ell$. Dually, $S, T \in \mathcal{F}_r$ if and only if $S^*, T^* \in \mathcal{F}_\ell$, which (as we have seen above) implies $T^* S^* \in \mathcal{F}_\ell$, which means that $ST = (T^* S^*)^* \in \mathcal{F}_r$.

(b) We shall split the proof of $\operatorname{ind}(ST) = \operatorname{ind}(S) + \operatorname{ind}(T)$ into three parts.

(b$_1$) Take $S, T \in \mathcal{F}_\ell$. First suppose $\operatorname{ind}(S)$ and $\operatorname{ind}(T)$ are both finite or, equivalently, suppose $S, T \in \mathcal{F} = \mathcal{F}_\ell \cap \mathcal{F}_r$. Consider the surjective transformation $L \colon \mathcal{N}(ST) \to \mathcal{R}(L) \subseteq \mathcal{H}$ defined by

$$Lx = Tx \quad \text{for every} \quad x \in \mathcal{N}(ST),$$

which is clearly linear with $\mathcal{N}(L) = \mathcal{N}(T) \cap \mathcal{N}(ST)$ by the very definition of L. Since $\mathcal{N}(T) \subseteq \mathcal{N}(ST)$, it follows that

$$\mathcal{N}(L) = \mathcal{N}(T).$$

If $y \in \mathcal{R}(L) = L(\mathcal{N}(ST))$, then $y = Lx = Tx$ for some vector $x \in \mathcal{H}$ such that $STx = 0$, and so $Sy = 0$, which implies that $y \in \mathcal{R}(T) \cap \mathcal{N}(S)$. Therefore, $\mathcal{R}(L) \subseteq \mathcal{R}(T) \cap \mathcal{N}(S)$. Conversely, if $y \in \mathcal{R}(T) \cap \mathcal{N}(S)$, then $y = Tx$ for some vector $x \in \mathcal{H}$ and $Sy = 0$, so that $STx = 0$, which implies that $x \in \mathcal{N}(ST)$ and $y = Tx = Lx \in \mathcal{R}(L)$. Thus $\mathcal{R}(T) \cap \mathcal{N}(S) \subseteq \mathcal{R}(L)$. Hence,

$$\mathcal{R}(L) = \mathcal{R}(T) \cap \mathcal{N}(S).$$

Again, as is well known from linear algebra, $\dim \mathcal{X} = \dim \mathcal{N}(L) + \dim \mathcal{R}(L)$ for every linear transformation L on a linear space $\mathcal{X}$ (see, e.g., [66, Problem 2.17]). Thus, with $\mathcal{X} = \mathcal{N}(ST)$, and recalling that $\dim \mathcal{N}(S) < \infty$,

$$\begin{aligned}
\dim \mathcal{N}(ST) &= \dim \mathcal{N}(T) + \dim \left(\mathcal{R}(T) \cap \mathcal{N}(S) \right) \\
&= \dim \mathcal{N}(T) + \dim \mathcal{N}(S) + \dim \left(\mathcal{R}(T) \cap \mathcal{N}(S) \right) - \dim \mathcal{N}(S).
\end{aligned}$$

Since $\mathcal{R}(T)$ is closed, $\mathcal{R}(T) \cap \mathcal{N}(S)$ is a subspace of $\mathcal{N}(S)$, so that (by the Projection Theorem) $\mathcal{N}(S) = \mathcal{R}(T) \cap \mathcal{N}(S) \oplus (\mathcal{N}(S) \ominus (\mathcal{R}(T) \cap \mathcal{N}(S)))$, and hence $\dim (\mathcal{N}(S) \ominus (\mathcal{R}(T) \cap \mathcal{N}(S))) = \dim \mathcal{N}(S) - \dim (\mathcal{R}(T) \cap \mathcal{N}(S))$ (because $\dim \mathcal{N}(S) < \infty$). Thus

$$\dim \mathcal{N}(T) + \dim \mathcal{N}(S) = \dim \mathcal{N}(ST) + \dim (\mathcal{N}(S) \ominus (\mathcal{R}(T) \cap \mathcal{N}(S))).$$

Swapping T with S^* and S with T^* (which have finite-dimensional kernels and closed ranges), it follows by Lemma 1.4 and Proposition 1.H(a,b) that

$$\begin{aligned}
\dim \mathcal{N}(S^*) + \dim \mathcal{N}(T^*) &= \dim \mathcal{N}(T^*S^*) + \dim(\mathcal{N}(T^*) \ominus (\mathcal{R}(S^*) \cap \mathcal{N}(T^*))) \\
&= \dim \mathcal{N}((ST)^*) + \dim(\mathcal{R}(T)^\perp \ominus (\mathcal{N}(S)^\perp \cap \mathcal{R}(T)^\perp)) \\
&= \dim \mathcal{N}((ST)^*) + \dim(\mathcal{R}(T)^\perp \ominus (\mathcal{R}(T) + \mathcal{N}(S))^\perp) \\
&= \dim \mathcal{N}((ST)^*) + \dim(\mathcal{N}(S) \ominus (\mathcal{R}(T) \cap \mathcal{N}(S))).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\dim \mathcal{N}(S) - \dim \mathcal{N}(S^*) &+ \dim \mathcal{N}(T) - \dim \mathcal{N}(T^*) \\
&= \dim \mathcal{N}(ST) - \dim \mathcal{N}((ST)^*),
\end{aligned}$$

and so
$$\mathrm{ind}\,(ST) = \mathrm{ind}\,(S) + \mathrm{ind}\,(T).$$

(b$_2$) Recall that if $S, T \in \mathcal{F}_\ell$, then $\mathrm{ind}\,(S), \mathrm{ind}\,(T) \neq +\infty$. Suppose $S, T \in \mathcal{F}_\ell$ and one of $\mathrm{ind}\,(S)$ or $\mathrm{ind}\,(T)$ is not finite, which means that $\mathrm{ind}\,(S) = -\infty$ or $\mathrm{ind}\,(T) = -\infty$. If $\mathrm{ind}\,(S) = -\infty$, then $\dim \mathcal{N}(S^*) = \infty$, and so $\dim \mathcal{R}(S)^\perp = \dim \mathcal{N}(S^*) = \infty$. Since $\mathcal{R}(ST) \subseteq \mathcal{R}(S)$, we get $\mathcal{R}(S)^\perp \subseteq \mathcal{R}(ST)^\perp$, and hence $\dim \mathcal{N}((TS)^*) = \dim \mathcal{R}(ST)^\perp = \infty$. Since $TS \in \mathcal{F}_\ell$ by item (a), Theorem 5.1 says that $\dim \mathcal{N}(TS) < \infty$. Thus $\mathrm{ind}\,(TS) = \dim \mathcal{N}(TS) - \dim \mathcal{N}((TS)^*) = -\infty$. On the other hand, if $\mathrm{ind}\,(S) \neq -\infty$, then $\mathrm{ind}\,(T) = -\infty$, and the same argument leads to $\mathrm{ind}\,(ST) = -\infty$. So, in both cases, $\mathrm{ind}\,(ST) = \mathrm{ind}\,(TS)$ and
$$\mathrm{ind}\,(ST) = -\infty = \mathrm{ind}\,(S) + \mathrm{ind}\,(T).$$

(b$_3$) Finally, suppose $S, T \in \mathcal{F}_r$. Thus $S^*, T^* \in \mathcal{F}_\ell$, which (as we have seen in (b$_1$) and (b$_2$)) implies that $\mathrm{ind}\,(T^*S^*) = \mathrm{ind}\,(S^*T^*) = \mathrm{ind}\,(S^*) + \mathrm{ind}\,(T^*)$. But $\mathrm{ind}\,(S^*) = -\mathrm{ind}\,(S)$, $\mathrm{ind}\,(T^*) = -\mathrm{ind}\,(T)$, $\mathrm{ind}\,(T^*S^*) = \mathrm{ind}\,((ST)^*) = -\mathrm{ind}\,(ST)$, and $\mathrm{ind}\,(S^*T^*) = \mathrm{ind}\,((TS)^*) = -\mathrm{ind}\,(TS)$. Thus, if $S, T \in \mathcal{F}_r$,
$$\mathrm{ind}\,(ST) = \mathrm{ind}\,(S) + \mathrm{ind}\,(T). \qquad \square$$

If $S, T \in \mathcal{SF} \backslash \mathcal{F}$, so that $\mathrm{ind}\,(S)$ and $\mathrm{ind}\,(T)$ are both not finite, then the expression $\mathrm{ind}\,(S) + \mathrm{ind}\,(T)$ makes sense as an extended integer in $\overline{\mathbb{Z}}$ if and only if either $\mathrm{ind}\,(S) = \mathrm{ind}\,(T) = +\infty$ or $\mathrm{ind}\,(S) = \mathrm{ind}\,(T) = -\infty$. But this is equivalent to saying that either $S, T \in \mathcal{F}_r \backslash \mathcal{F} = \mathcal{F}_r \backslash \mathcal{F}_\ell$ or $S, T \in \mathcal{F}_\ell \backslash \mathcal{F} = \mathcal{F}_\ell \backslash \mathcal{F}_r$. Therefore, if $S, T \in \mathcal{SF}$ with one of them in $\mathcal{F}_r \backslash \mathcal{F}_\ell$ and the other in $\mathcal{F}_\ell \backslash \mathcal{F}_r$, then the index expression of Theorem 5.4 does not make sense. Moreover, if one of them lies in $\mathcal{F}_r \backslash \mathcal{F}_\ell$ and the other lies in $\mathcal{F}_\ell \backslash \mathcal{F}_r$, then it may happen that their product is not in $\mathcal{SF}$. For instance, let $\mathcal{H}$ be an infinite-dimensional Hilbert space, and let S_+ be the canonical unilateral shift (of infinite multiplicity) on the Hilbert space $\ell^2_+(\mathcal{H}) = \bigoplus_{k=0}^\infty \mathcal{H}$ (cf. Section 2.7). It is readily verified that $\dim \mathcal{N}(S_+) = \infty$, $\dim \mathcal{N}(S_+^*) = 0$, and $\mathcal{R}(S_+)$ is closed in $\ell^2_+(\mathcal{H})$. In fact, $\mathcal{N}(S_+) = \ell^2_+(\mathcal{H}) \ominus \bigoplus_{k=1}^\infty \mathcal{H} \cong \mathcal{H}$, $\mathcal{N}(S_+^*) = \{0\}$, and $\mathcal{R}(S_+) = \{0\} \oplus \bigoplus_{k=1}^\infty \mathcal{H}$. Therefore, $S_+ \in \mathcal{F}_r \backslash \mathcal{F}_\ell$ and $S_+^* \in \mathcal{F}_\ell \backslash \mathcal{F}_r$ (according to Theorem 5.1). However, it is easy to see that $S_+ S_+^* \notin \mathcal{SF}$. Indeed, $S_+ S_+^* = O \oplus I$ (where O stands for the null operator on $\mathcal{H}$ and I for the identity operator on $\bigoplus_{k=1}^\infty \mathcal{H}$) does not lie in $\mathcal{SF}$ because it is self-adjoint and $\dim \mathcal{N}(S_+ S_+^*) = \infty$ (cf. Theorem 5.1 again).

Corollary 5.5. *Take any nonnegative integer n. If $T \in \mathcal{F}_\ell$ (or $T \in \mathcal{F}_r$), then $T^n \in \mathcal{F}_\ell$ (or $T^n \in \mathcal{F}_r$) and*
$$\mathrm{ind}\,(T^n) = n\,\mathrm{ind}\,(T).$$

Proof. The result holds trivially for $n = 0$ (the identity is Fredholm with index zero), and tautologically for $n = 1$. Thus suppose $n \geq 2$. Theorem 5.4 says

that the claimed result holds for $n = 2$. By using Theorem 5.4 again, a trivial induction ensures that the claimed result holds for each $n \geq 2$. $\qquad\square$

The null operator on an infinite-dimensional space is not Fredholm, and so a compact operator may not be Fredholm. However, the sum of a Fredholm operator and a compact operator is a Fredholm operator with the same index. In other words, the index of a Fredholm operator remains unchanged under compact perturbation, which is referred to as the *index stability*.

Theorem 5.6. *Take $T \in \mathcal{B}[\mathcal{H}]$ and $K \in \mathcal{B}_\infty[\mathcal{H}]$.*

(a) $T \in \mathcal{F}_\ell \iff T + K \in \mathcal{F}_\ell$ *and* $T \in \mathcal{F}_r \iff T + K \in \mathcal{F}_r$.

   In particular,

$$T \in \mathcal{F} \iff T + K \in \mathcal{F} \quad \text{and} \quad T \in \mathcal{SF} \iff T + K \in \mathcal{SF}.$$

(b) *Moreover, for every $T \in \mathcal{SF}$,*

$$\mathrm{ind}\,(T + K) = \mathrm{ind}\,(T).$$

Proof. (a) If $T \in \mathcal{F}_\ell$, then there exist $A \in \mathcal{B}[\mathcal{H}]$ and $K_1 \in \mathcal{B}_\infty[\mathcal{H}]$ such that $AT = I + K_1$. So $A \in \mathcal{F}_r$. Take an arbitrary $K \in \mathcal{B}_\infty[\mathcal{H}]$. Set $K_2 = K_1 + AK$, which lies in $\mathcal{B}_\infty[\mathcal{H}]$ because $\mathcal{B}_\infty[\mathcal{H}]$ is an ideal of $\mathcal{B}[\mathcal{H}]$. Since $A(T + K) = I + K_2$, it follows that $T + K \in \mathcal{F}_\ell$. Therefore, $T \in \mathcal{F}_\ell$ implies $T + K \in \mathcal{F}_\ell$. The converse also holds because $T = (T + K) - K$. Hence

$$T \in \mathcal{F}_\ell \quad \iff \quad T + K \in \mathcal{F}_\ell.$$

Dually, if $T \in \mathcal{F}_r$ and $K \in \mathcal{B}_\infty[\mathcal{H}]$, then $T^* \in \mathcal{F}_\ell$ and $K^* \in \mathcal{B}_\infty[\mathcal{H}]$, so that $T^* + K^* \in \mathcal{F}_\ell$, and hence $T + K = (T^* + K^*)^* \in \mathcal{F}_r$. Therefore,

$$T \in \mathcal{F}_r \quad \iff \quad T + K \in \mathcal{F}_r.$$

(b) First suppose $T \in \mathcal{F}_\ell$. The Fredholm Alternative of Corollary 1.20 says that $I + K_1$ and $I + K_2$ are Fredholm operators with Fredholm indices zero (see Remark 5.3(d)). Therefore, $AT = I + K_1 \in \mathcal{F}$ with $\mathrm{ind}\,(AT) = 0$ and $A(T + K) = I + K_2 \in \mathcal{F}$ with $\mathrm{ind}\,(A(T + K)) = 0$. Note that, since $T \in \mathcal{F}_\ell$, then $A \in \mathcal{F}_r$. If $T \in \mathcal{F}$, then $T + K \in \mathcal{F}$ by item (a). In this case, applying Theorem 5.4 for operators in $\mathcal{F}_r$, we get

$$\mathrm{ind}\,(T + K) + \mathrm{ind}\,(A) = \mathrm{ind}\,(A(T + K)) = 0 = \mathrm{ind}\,(AT) = \mathrm{ind}\,(A) + \mathrm{ind}\,(T).$$

Since $\mathrm{ind}\,(T)$ is finite (under the assumption that $T \in \mathcal{F}$), it follows that $\mathrm{ind}\,(A) = -\mathrm{ind}\,(T)$ is finite as well, and so we may subtract $\mathrm{ind}\,(A)$ to get

$$\mathrm{ind}\,(T + K) = \mathrm{ind}\,(T).$$

On the other hand, if $T \in \mathcal{F}_\ell \backslash \mathcal{F} = \mathcal{F}_\ell \backslash \mathcal{F}_r$, then $T + K \in \mathcal{F}_\ell \backslash \mathcal{F}_r$ by (a). In this case (see Remark 5.3(e)),

$$\operatorname{ind}(T + K) = -\infty = \operatorname{ind}(T).$$

Dually, if $T \in \mathcal{F}_r \backslash \mathcal{F}$ and $K \in \mathcal{B}_\infty[\mathcal{H}]$, then $T^* \in \mathcal{F}_\ell \backslash \mathcal{F}$ and $K^* \in \mathcal{B}_\infty[\mathcal{H}]$. So

$$\operatorname{ind}(T + K) = -\operatorname{ind}(T^* + K^*) = -\operatorname{ind}(T^*) = \operatorname{ind}(T). \qquad \square$$

Remark 5.7. (a) INDEX OF A. We have established the following assertion in the preceding proof.

> *If $T \in \mathcal{F}$ and $A \in \mathcal{B}[\mathcal{H}]$ is such that $AT = I + K$ for some $K \in \mathcal{B}_\infty[\mathcal{H}]$, then $A \in \mathcal{F}$ and $\operatorname{ind}(A) = -\operatorname{ind}(T) = \operatorname{ind}(T^*)$.*

Moreover, if $T \in \mathcal{F}$, then $T \in \mathcal{F}_r$, so that $TA = I + K$ for some $K \in \mathcal{B}_\infty[\mathcal{H}]$. Thus $A \in \mathcal{F}_\ell$. Therefore, a similar argument leads to the dual statement.

> *If $T \in \mathcal{F}$ and $A \in \mathcal{B}[\mathcal{H}]$ is such that $TA = I + K$ for some $K \in \mathcal{B}_\infty[\mathcal{H}]$, then $A \in \mathcal{F}$ and $\operatorname{ind}(A) = -\operatorname{ind}(T) = \operatorname{ind}(T^*)$.*

(b) WEYL OPERATOR. A *Weyl operator* is a Fredholm operator with null index (equivalently, a semi-Fredholm operator with null index). Let

$$\mathcal{W} = \{T \in \mathcal{F} : \operatorname{ind}(T) = 0\}$$

denote the class of all Weyl operators from $\mathcal{B}[\mathcal{H}]$. Since $T \in \mathcal{F}$ if and only if $T^* \in \mathcal{F}$ and $\operatorname{ind}(T^*) = -\operatorname{ind}(T)$, it follows that

$$T \in \mathcal{W} \quad \Longleftrightarrow \quad T^* \in \mathcal{W}.$$

Items (c), (d), and (f) in Remark 5.3 ensure that (i) every operator on a finite-dimensional space is a Weyl operator,

$$\dim \mathcal{H} < \infty \quad \Longrightarrow \quad \mathcal{W} = \mathcal{B}[\mathcal{H}],$$

(ii) the Fredholm Alternative can be rephrased as

$$K \in \mathcal{B}_\infty[\mathcal{H}] \text{ and } \lambda \neq 0 \quad \Longrightarrow \quad \lambda I - K \in \mathcal{W},$$

and (iii) every nonzero multiple of a Weyl operator is again a Weyl operator,

$$T \in \mathcal{W} \quad \Longrightarrow \quad \gamma T \in \mathcal{W} \text{ for every } \gamma \in \mathbb{C} \backslash \{0\}.$$

In particular, every nonzero scalar operator is a Weyl operator. In fact, the product of two Weyl operators is again a Weyl operator (by Theorem 5.4),

$$S, T \in \mathcal{W} \quad \Longrightarrow \quad ST \in \mathcal{W}.$$

Thus integral powers of Weyl operators are Weyl operators (Corollary 5.5),

$$T \in \mathcal{W} \quad \Longrightarrow \quad T^n \in \mathcal{W} \text{ for every } n \in \mathbb{N}_0.$$

On an infinite-dimensional space, the identity is Weyl but not compact; and the null operator is compact but not semi-Fredholm. Actually,

$$T \in \mathcal{F} \cap \mathcal{B}_\infty[\mathcal{H}] \quad \Longleftrightarrow \quad \dim \mathcal{H} < \infty \quad (\Longleftrightarrow \quad T \in \mathcal{W} \cap \mathcal{B}_\infty[\mathcal{H}]\ ).$$

(Set $T = -K$ in Theorem 5.6 and, for the converse, see Remark 5.3(c).) Also note that, if T is normal, then $\mathcal{N}(T) = \mathcal{N}(T^*T) = \mathcal{N}(TT^*) = \mathcal{N}(T^*)$ by Lemma 1.4, and so every normal Fredholm operator is Weyl,

$$T \text{ normal in } \mathcal{F} \quad \Longrightarrow \quad T \in \mathcal{W}.$$

Since $T \in \mathcal{B}[\mathcal{H}]$ is invertible if and only $\mathcal{N}(T) = \{0\}$ and $\mathcal{R}(T) = \mathcal{H}$ (Theorem 1.1), which happens if and only if $T^* \in \mathcal{B}[\mathcal{H}]$ is invertible (Proposition 1.L), it follows by Corollary 5.2 that every invertible operator is Weyl,

$$T \in \mathcal{G}[\mathcal{H}] \quad \Longrightarrow \quad T \in \mathcal{W}.$$

Theorem 5.6 ensures that, for every compact $K \in \mathcal{B}_\infty[\mathcal{H}]$,

$$T \in \mathcal{W} \quad \Longrightarrow \quad T + K \in \mathcal{W}.$$

5.2 Essential Spectrum and Spectral Picture

An element a in a unital algebra $\mathcal{A}$ is *left invertible* if there is an element a_ℓ in $\mathcal{A}$ (a *left inverse* of a) such that $a_\ell a = 1$, where 1 stands for the identity in $\mathcal{A}$, and it is *right invertible* if there is an element a_r in $\mathcal{A}$ (a *right inverse* of a) such that $a a_r = 1$. An element a in $\mathcal{A}$ is *invertible* if there is an element a^{-1} in $\mathcal{A}$ (the *inverse* of a) such that $a^{-1}a = a a^{-1} = 1$. Thus a in $\mathcal{A}$ is invertible if and only if it has a left inverse a_ℓ in $\mathcal{A}$ and a right inverse a_r in $\mathcal{A}$, which coincide with its inverse a^{-1} in $\mathcal{A}$ (since $a_r = a_\ell a a_r = a_\ell$).

Lemma 5.8. *Take an operator S in the unital Banach algebra $\mathcal{B}[\mathcal{H}]$.*

(a) *S is left invertible if and only if it is injective with a closed range (i.e., $\mathcal{N}(S) = \{0\}$ and $\mathcal{R}(S)^- = \mathcal{R}(S)$).*

(b) *S is left (right) invertible if and only if S^* is right (left) invertible.*

(c) *S is right invertible if and only if it is surjective (i.e., $\mathcal{R}(S) = \mathcal{H}$).*

Proof. $S \in \mathcal{B}[\mathcal{H}]$ is said to have a bounded inverse on its range if there exists $S^{-1} \in \mathcal{B}[\mathcal{R}(S), \mathcal{H}]$ such that $S^{-1}S = I$ in $\mathcal{B}[\mathcal{H}]$ and $S S^{-1} = I$ in $\mathcal{B}[\mathcal{R}(S)]$).

Claim. S is left invertible if and only if it has a bounded inverse on its range.

Proof. If $S_\ell \in \mathcal{B}[\mathcal{H}]$ is such that $S_\ell S = I$, then $S_\ell|_{\mathcal{R}(S)} \in \mathcal{B}[\mathcal{R}(S), \mathcal{H}]$ is such that $S_\ell|_{\mathcal{R}(S)} S = I \in \mathcal{B}[\mathcal{H}]$. Consider the operator $S S_\ell|_{\mathcal{R}(S)}$ in $\mathcal{B}[\mathcal{R}(S)]$, and take any y in $\mathcal{R}(S)$. Thus $y = Sx$ for some x in $\mathcal{H}$, and so $S_\ell|_{\mathcal{R}(S)} y = S_\ell Sx =$

x. Then $SS_\ell|_{\mathcal{R}(S)}y = Sx = y$, and hence $SS_\ell|_{\mathcal{R}(S)} = I \in \mathcal{B}[\mathcal{R}(S)]$. Set $S^{-1} = S_\ell|_{\mathcal{R}(S)} \in \mathcal{B}[\mathcal{R}(S), \mathcal{H}]$, and get $S^{-1}S = I$ in $\mathcal{B}[\mathcal{H}]$ and $SS^{-1} = I$ in $\mathcal{B}[\mathcal{R}(S)]$. Conversely, any extension of $S^{-1} \in \mathcal{B}[\mathcal{R}(S), \mathcal{H}]$ over the whole space $\mathcal{H}$ (cf. Proposition 1.G(b)) is a left inverse of $S \in \mathcal{B}[\mathcal{H}]$. This proves the Claim.

(a) But S has a bounded inverse on its range if and only if it is injective (i.e., $\mathcal{N}(S) = \{0\}$) and $\mathcal{R}(S)$ is closed, according to Theorem 1.2.

(b) Since $S_\ell S = I$ ($SS_r = I$) if and only if $S^*S_\ell^* = I$ ($S_r^*S^* = I$), it follows that S is left (right) invertible if and only if S^* is right (left) invertible.

(c) Thus we get from items (a) and (b) that S is right invertible if and only if $\mathcal{N}(S^*) = \{0\}$ and $\mathcal{R}(S^*)$ is closed, or equivalently (cf. Lemmas 1.4 and 1.5), $\mathcal{R}(S) = \mathcal{R}(S)^- = \mathcal{H}$; which means that $\mathcal{R}(S) = \mathcal{H}$. $\qquad\square$

Let T be an operator in $\mathcal{B}[\mathcal{H}]$. The *left spectrum* $\sigma_\ell(T)$ and the *right spectrum* $\sigma_r(T)$ of $T \in \mathcal{B}[\mathcal{H}]$ are the sets

$$\sigma_\ell(T) = \{\lambda \in \mathbb{C} \colon \lambda I - T \text{ is not left invertible}\},$$

$$\sigma_r(T) = \{\lambda \in \mathbb{C} \colon \lambda I - T \text{ is not right invertible}\},$$

so that the spectrum $\sigma(T)$ of $T \in \mathcal{B}[\mathcal{H}]$ is given by

$$\sigma(T) = \{\lambda \in \mathbb{C} \colon \lambda I - T \text{ is not invertible}\} = \sigma_\ell(T) \cup \sigma_r(T).$$

Corollary 5.9. *Take an operator* $T \in \mathcal{B}[\mathcal{H}]$.

$$\sigma_\ell(T) = \{\lambda \in \mathbb{C} \colon \mathcal{R}(\lambda I - T) \text{ is not closed } \text{ or } \mathcal{N}(\lambda I - T) \neq \{0\}\},$$

$$\sigma_r(T) = \{\lambda \in \mathbb{C} \colon \mathcal{R}(\lambda I - T) \text{ is not closed } \text{ or } \mathcal{N}(\overline{\lambda}I - T^*) \neq \{0\}\}$$
$$= \{\lambda \in \mathbb{C} \colon \mathcal{R}(\lambda I - T) \text{ is not closed } \text{ or } \mathcal{R}(\lambda I - T)^- \neq \mathcal{H}\}$$
$$= \{\lambda \in \mathbb{C} \colon \mathcal{R}(\lambda I - T) \neq \mathcal{H}\}.$$

Proof. Take $T \in \mathcal{B}[\mathcal{H}]$ and $\lambda \in \mathbb{C}$. Set $S = \lambda I - T$ in $\mathcal{B}[\mathcal{H}]$. By Lemma 5.8, S is not left invertible if and only if $\mathcal{R}(S)$ is not closed or S is not injective (i.e., or $\mathcal{N}(S) \neq 0$). This proves the expression for $\sigma_\ell(T)$. By Lemma 5.8, S is not right invertible if and only if $\mathcal{R}(S) \neq \mathcal{H}$, which means that $\mathcal{R}(S) \neq \mathcal{R}(S)^-$ or $\mathcal{R}(S)^- \neq \mathcal{H}$, which in turn is equivalent to saying that $\mathcal{R}(S) \neq \mathcal{R}(S)^-$ or $\mathcal{N}(S^*) \neq \{0\}$ (Lemma 1.5), thus proving the expressions for $\sigma_r(T)$. $\qquad\square$

By Corollary 5.9 and Theorem 2.6 (see also the diagram of Section 2.2),

$$\sigma_\ell(T) = \sigma(T) \backslash \sigma_{R_1}(T) = \sigma_{AP}(T), \qquad \sigma_r(T) = \sigma(T) \backslash \sigma_{P_1}(T) = \sigma_{AP}(T^*)^*,$$

and so

$$\sigma_\ell(T) \text{ and } \sigma_r(T) \text{ are closed subsets of } \mathbb{C}.$$

Indeed, $\sigma_{AP}(T)$ is closed (cf. Theorem 2.5). Note that $\sigma_{P_1}(T)$ and $\sigma_{R_1}(T)$ are open in $\mathbb{C}$ (cf. Remarks following Theorems 2.5 and 2.6), and therefore $\sigma_{AP}(T) = \sigma(T) \backslash \sigma_{R_1}(T) = \sigma(T) \cap (\mathbb{C} \backslash \sigma_{R_1}(T))$ and $\sigma_{AP}(T^*)^* = \sigma(T) \backslash \sigma_{P_1}(T) = \sigma(T) \cap (\mathbb{C} \backslash \sigma_{P_1}(T))$ are closed (and bounded) in $\mathbb{C}$, since $\sigma(T)$ is closed (and bounded); and so is the intersection

$$\sigma_\ell(T) \cap \sigma_r(T) = \sigma(T) \backslash (\sigma_{P_1}(T) \cup \sigma_{R_1}(T)) = \sigma_{AP}(T) \cap \sigma_{AP}(T^*)^*.$$

The fact that each $\sigma_{P_1}(T)$ and $\sigma_{R_1}(T)$ is an open subset of $\mathbb{C}$, and so each $\sigma_\ell(T)$ and $\sigma_r(T)$ is a closed subset of $\mathbb{C}$, it a consequence of the property that T lies in the unital Banach algebra $\mathcal{B}[\mathcal{H}]$ (so that $\sigma(T)$ is a compact set), which actually happens in any unital Banach algebra. Also recall that $(S_\ell S)^* = S^* S_\ell^*$ and $(S S_r)^* = S_r^* S^*$, so that S_ℓ is a left inverse of S if and only if S_ℓ^* is a right inverse of S^*, and S_r is a right inverse for S if and only if S_r^* is a left inverse for S^*. Therefore (as we can also infer from Corollary 5.9),

$$\sigma_\ell(T) = \sigma_r(T^*)^* \quad \text{and} \quad \sigma_r(T) = \sigma_\ell(T^*)^*.$$

Consider the *Calkin algebra* $\mathcal{B}[\mathcal{H}]/\mathcal{B}_\infty[\mathcal{H}]$ — the quotient algebra of $\mathcal{B}[\mathcal{H}]$ modulo the ideal $\mathcal{B}_\infty[\mathcal{H}]$ of all compact operators. If $\dim \mathcal{H} < \infty$, then all operators are compact, so that $\mathcal{B}[\mathcal{H}]/\mathcal{B}_\infty[\mathcal{H}]$ is trivially null. Thus, whenever the Calkin algebra is brought into play, it will be assumed that $\dim \mathcal{H} = \infty$. Since $\mathcal{B}_\infty[\mathcal{H}]$ is a subspace of $\mathcal{B}[\mathcal{H}]$, it follows that $\mathcal{B}[\mathcal{H}]/\mathcal{B}_\infty[\mathcal{H}]$ is a *unital Banach algebra* whenever $\mathcal{H}$ is *infinite dimensional*. Define the *natural map* (or the *natural quotient map*) $\pi \colon \mathcal{B}[\mathcal{H}] \to \mathcal{B}[\mathcal{H}]/\mathcal{B}_\infty[\mathcal{H}]$ by

$$\pi(T) = [T] = \{S \in \mathcal{B}[\mathcal{H}] \colon S = T + K \text{ for some } K \in \mathcal{B}_\infty[\mathcal{H}]\} = T + \mathcal{B}_\infty[\mathcal{H}]$$

for every T in $\mathcal{B}[\mathcal{H}]$. Recall that the origin of the linear space $\mathcal{B}[\mathcal{H}]/\mathcal{B}_\infty[\mathcal{H}]$ is $[0] = \mathcal{B}_\infty[\mathcal{H}] \in \mathcal{B}[\mathcal{H}]/\mathcal{B}_\infty[\mathcal{H}]$, the kernel of the natural map π is

$$\mathcal{N}(\pi) = \{T \in \mathcal{B}[\mathcal{H}] \colon \pi(T) = [0]\} = \mathcal{B}_\infty[\mathcal{H}] \subseteq \mathcal{B}[\mathcal{H}],$$

and π is a *unital homomorphism*. Indeed, since $\mathcal{B}_\infty[\mathcal{H}]$ is an ideal of $\mathcal{B}[\mathcal{H}]$,

$$\pi(T + T') = (T + T') + \mathcal{B}_\infty[\mathcal{H}] = (T + \mathcal{B}_\infty[\mathcal{H}]) + (T' + \mathcal{B}_\infty[\mathcal{H}]) = \pi(T) + \pi(T'),$$

$$\pi(T T') = (T T') + \mathcal{B}_\infty[\mathcal{H}] = (T + \mathcal{B}_\infty[\mathcal{H}])(T' + \mathcal{B}_\infty[\mathcal{H}]) = \pi(T)\pi(T'),$$

for every $T, T' \in \mathcal{B}[\mathcal{H}]$, and $\pi(I) = [I]$ is the identity element of the algebra $\mathcal{B}[\mathcal{H}]/\mathcal{B}_\infty[\mathcal{H}]$. Moreover, the norm on $\mathcal{B}[\mathcal{H}]/\mathcal{B}_\infty[\mathcal{H}]$ is given by

$$\|[T]\| = \inf_{K \in \mathcal{B}_\infty[\mathcal{H}]} \|T + K\| \leq \|T\|,$$

so that π is a contraction.

Theorem 5.10. *Take any operator* $T \in \mathcal{B}[\mathcal{H}]$. $T \in \mathcal{F}_\ell$ *(or* $T \in \mathcal{F}_r$*) if and only if* $\pi(T)$ *is left (or right) invertible in the Calkin algebra* $\mathcal{B}[\mathcal{H}]/\mathcal{B}_\infty[\mathcal{H}]$.

Proof. We show that the following assertions are pairwise equivalent.

(a) $T \in \mathcal{F}_\ell$.

(b) There is a pair of operators $\{A, K\}$ with $A \in \mathcal{B}[\mathcal{H}]$ and $K \in \mathcal{B}_\infty[\mathcal{H}]$ such that $AT = I + K$.

(c) There exists a quadruple of operators $\{A, K_1, K_2, K_3\}$ with $A \in \mathcal{B}[\mathcal{H}]$ and $K_1, K_2, K_3 \in \mathcal{B}_\infty[\mathcal{H}]$ such that $(A + K_1)(T + K_2) = I + K_3$.

(d) $\pi(A)\pi(T) = \pi(I)$, identity in $\mathcal{B}[\mathcal{H}]/\mathcal{B}_\infty[\mathcal{H}]$, for some $\pi(A)$ in $\mathcal{B}[\mathcal{H}]/\mathcal{B}_\infty[\mathcal{H}]$.

(e) $\pi(T)$ is left invertible in $\mathcal{B}[\mathcal{H}]/\mathcal{B}_\infty[\mathcal{H}]$.

By definition, (a) and (b) are equivalent, and (b) implies (c) trivially. If (c) holds, then there exist $B \in [A] = \pi(A)$, $S \in [T] = \pi(T)$, and $J \in [I] = \pi(I)$ such that $BS = J$. Therefore, $\pi(A)\pi(T) = [A][T] = [B][S] = \pi(B)\pi(S) = \pi(BS) = \pi(J) = [J] = [I] = \pi(I)$, so that (c) implies (d). Now observe that $X \in \pi(A)\pi(T) = \pi(AT)$ if and only if $X = AT + K_1$ for some $K_1 \in \mathcal{B}_\infty[\mathcal{H}]$, and $X \in \pi(I)$ if and only if $X = I + K_2$ for some $K_2 \in \mathcal{B}_\infty[\mathcal{H}]$. Therefore, if $\pi(A)\pi(T) = \pi(I)$, then $X = AT + K_1$ if and only if $X = I + K_2$, and hence $AT = I + K$ with $K = K_2 - K_1 \in \mathcal{B}_\infty[\mathcal{H}]$. Thus (d) implies (b). Finally, (d) and (e) are equivalent by the definition of left invertibility.

Outcome: $T \in \mathcal{F}_\ell$ if and only if $\pi(T)$ is left invertible in $\mathcal{B}[\mathcal{H}]/\mathcal{B}_\infty[\mathcal{H}]$. Dually, $T \in \mathcal{F}_r$ if and only if $\pi(T)$ is right invertible in $\mathcal{B}[\mathcal{H}]/\mathcal{B}_\infty[\mathcal{H}]$. □

The *essential spectrum* (or the *Calkin spectrum*) $\sigma_e(T)$ of $T \in \mathcal{B}[\mathcal{H}]$ is the spectrum of $\pi(T)$ in unital Banach algebra $\mathcal{B}[\mathcal{H}]/\mathcal{B}_\infty[\mathcal{H}]$,

$$\sigma_e(T) = \sigma(\pi(T)),$$

and so $\sigma_e(T)$ is a compact subset of $\mathbb{C}$. Similarly, the *left essential spectrum* $\sigma_{\ell e}(T)$ and the *right essential spectrum* $\sigma_{re}(T)$ of $T \in \mathcal{B}[\mathcal{H}]$ are defined as the left and the right spectrum of $\pi(T)$ in the Calkin algebra $\mathcal{B}[\mathcal{H}]/\mathcal{B}_\infty[\mathcal{H}]$:

$$\sigma_{\ell e}(T) = \sigma_\ell(\pi(T)) \quad \text{and} \quad \sigma_{re}(T) = \sigma_r(\pi(T)),$$

and so

$$\sigma_e(T) = \sigma_{\ell e}(T) \cup \sigma_{re}(T).$$

By using only the definitions of left and right semi-Fredholm operators, and of left and right essential spectra, we get the following characterization.

Corollary 5.11. *If $T \in \mathcal{B}[\mathcal{H}]$, then*

$$\sigma_{\ell e}(T) = \{\lambda \in \mathbb{C} \colon \lambda I - T \in \mathcal{B}[\mathcal{H}] \backslash \mathcal{F}_\ell\},$$

$$\sigma_{re}(T) = \{\lambda \in \mathbb{C} \colon \lambda I - T \in \mathcal{B}[\mathcal{H}] \backslash \mathcal{F}_r\}.$$

Proof. Take $T \in \mathcal{B}[\mathcal{H}]$. Theorem 5.10 says that $\lambda I - T \notin \mathcal{F}_\ell$ if and only if $\pi(\lambda I - T)$ is not left invertible in $\mathcal{B}[\mathcal{H}]/\mathcal{B}_\infty[\mathcal{H}]$, which means (definition of left spectrum) that $\lambda \in \sigma_\ell(\pi(T))$. But $\sigma_{\ell e}(T) = \sigma_\ell(\pi(T))$ — definition of left essential spectrum. Thus $\lambda \in \sigma_{\ell e}(T)$ if and only if $\lambda I - T \notin \mathcal{F}_\ell$. Dually, $\lambda \in \sigma_{re}(T)$ if and only if $\lambda I - T \notin \mathcal{F}_r$. □

Corollary 5.12. (ATKINSON THEOREM).

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : \lambda I - T \in \mathcal{B}[\mathcal{H}] \backslash \mathcal{F}\}.$$

Proof. The expressions for $\sigma_{\ell e}(T)$ and $\sigma_{re}(T)$ in Corollary 5.11 lead to the claimed identity, since $\sigma_e(T) = \sigma_{\ell e}(T) \cup \sigma_{re}(T)$ and $\mathcal{F} = \mathcal{F}_\ell \cap \mathcal{F}_r$. □

Thus the essential spectrum is the set of all scalars λ for which $\lambda I - T$ is not Fredholm. The following equivalent version is also frequently used [7].

Corollary 5.13. (ATKINSON THEOREM). *An operator $T \in \mathcal{B}[\mathcal{H}]$ is Fredholm if and only if its image $\pi(T)$ in the Calkin algebra $\mathcal{B}[\mathcal{H}]/\mathcal{B}_\infty[\mathcal{H}]$ is invertible.*

Proof. Straightforward from Theorem 5.10: T is Fredholm if and only if it is both left and right semi-Fredholm, and $\pi(T)$ is invertible in $\mathcal{B}[\mathcal{H}]/\mathcal{B}_\infty[\mathcal{H}]$ if and only if it is both left and right invertible in $\mathcal{B}[\mathcal{H}]/\mathcal{B}_\infty[\mathcal{H}]$. □

This is usually referred to by saying that T is Fredholm if and only if T is *essentially invertible*. Thus the essential spectrum $\sigma_e(T)$ is the set of all scalars λ for which $\lambda I - T$ is not essentially invertible (i.e., $\lambda I - T$ is not Fredholm), and so the essential spectrum is also called the *Fredholm spectrum*.

We have already seen that, in the unital Banach algebra $\mathcal{B}[\mathcal{H}]$, the sets $\sigma_\ell(T)$ and $\sigma_r(T)$ are closed in $\mathbb{C}$, because $\sigma(T)$ is a compact subset of $\mathbb{C}$. Similarly, in the unital Banach algebra $\mathcal{B}[\mathcal{H}]/\mathcal{B}_\infty[\mathcal{H}]$,

$$\sigma_{\ell e}(T) \text{ and } \sigma_{re}(T) \text{ are closed subsets of } \mathbb{C},$$

since $\sigma_e(T) = \sigma(\pi(T))$ is a compact set in $\mathbb{C}$. From Corollary 5.11 we get

$$\mathbb{C} \backslash \sigma_{\ell e}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \in \mathcal{F}_\ell\},$$

$$\mathbb{C} \backslash \sigma_{re}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \in \mathcal{F}_r\},$$

which are open subsets of $\mathbb{C}$. Thus, since $\mathcal{SF} = \mathcal{F}_\ell \cup \mathcal{F}_r$, it follows that

$$\mathbb{C} \backslash \big(\sigma_{\ell e}(T) \cap \sigma_{re}(T)\big) = \big(\mathbb{C} \backslash \sigma_{\ell e}(T)\big) \cup \big(\mathbb{C} \backslash \sigma_{re}(T)\big) = \{\lambda \in \mathbb{C} : \lambda I - T \in \mathcal{SF}\},$$

which is again an open subset of $\mathbb{C}$. Therefore, the intersection

$$\sigma_{\ell e}(T) \cap \sigma_{re}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \in \mathcal{B}[\mathcal{H}] \backslash \mathcal{SF}\}$$

is a closed subset of $\mathbb{C}$. Observe that, since $\mathcal{F} = \mathcal{F}_\ell \cap \mathcal{F}_r$, the complement of the union $\sigma_{\ell e}(T) \cup \sigma_{re}(T)$ is given by (cf. Corollary 5.12)

$$\mathbb{C}\backslash\sigma_e(T) = \mathbb{C}\backslash\big(\sigma_{\ell e}(T) \cup \sigma_{re}(T)\big)$$
$$= \big(\mathbb{C}\backslash\sigma_{\ell e}(T)\big) \cap \big(\mathbb{C}\backslash\sigma_{re}(T)\big) = \{\lambda \in \mathbb{C}\colon \lambda I - T \in \mathcal{F}\},$$

which is an open subset of $\mathbb{C}$.

Corollary 5.14. *If $T \in \mathcal{B}[\mathcal{H}]$, then*

$$\sigma_{\ell e}(T) = \big\{\lambda \in \mathbb{C}\colon \mathcal{R}(\lambda I - T) \ \textit{is not closed} \ \ \textit{or} \ \ \dim\mathcal{N}(\lambda I - T) = \infty\big\},$$

$$\sigma_{re}(T) = \big\{\lambda \in \mathbb{C}\colon \mathcal{R}(\lambda I - T) \ \textit{is not closed} \ \ \textit{or} \ \ \dim\mathcal{N}(\overline{\lambda} I - T^*) = \infty\big\}.$$

Proof. Theorem 5.1 and Corollary 5.11. $\square$

Note that, by Corollaries 5.9 and 5.14,

$$\sigma_{\ell e}(T) \subseteq \sigma_\ell(T) \quad \text{and} \quad \sigma_{re}(T) \subseteq \sigma_r(T)$$

and so

$$\sigma_e(T) \subseteq \sigma(T).$$

Corollary 5.14 also shows that

$$\sigma_{\ell e}(T) = \sigma_{re}(T^*)^* \quad \text{and} \quad \sigma_{re}(T) = \sigma_{\ell e}(T^*)^*,$$

and hence

$$\sigma_e(T) = \sigma_e(T^*)^*.$$

Moreover, by the previous results and the diagram of Section 2.2,

$$\sigma_C(T) \subseteq \sigma_{\ell e}(T) \subseteq \sigma_\ell(T) = \sigma(T)\backslash\sigma_{R_1}(T) = \sigma_{AP}(T),$$

$$\sigma_C(T) = \sigma_C(T^*)^* \subseteq \sigma_{\ell e}(T^*)^* = \sigma_{re}(T) \subseteq \sigma_r(T) = \sigma(T)\backslash\sigma_{P_1}(T) = \sigma_{AP}(T^*)^*,$$

so that

$$\sigma_{\ell e}(T) \cap \sigma_{re}(T) \subseteq \sigma(T)\backslash\big(\sigma_{P_1}(T) \cup \sigma_{R_1}(T)\big),$$

which is an inclusion of closed sets, and therefore

$$\sigma_{P_1}(T) \cup \sigma_{R_1}(T) \subseteq \sigma(T)\backslash\big(\sigma_{\ell e}(T) \cap \sigma_{re}(T)\big) \subseteq \mathbb{C}\backslash\big(\sigma_{\ell e}(T) \cap \sigma_{re}(T)\big),$$

which is an inclusion of open sets.

Remark 5.15. (a) FINITE DIMENSIONAL. For any $T \in \mathcal{B}[\mathcal{H}]$,

$$\sigma_e(T) \neq \varnothing \quad \Longleftrightarrow \quad \dim\mathcal{H} = \infty.$$

Actually, since $\sigma_e(T)$ was defined as the spectrum of $\pi(T)$ in the Calkin algebra $\mathcal{B}[\mathcal{H}]/\mathcal{B}_\infty[\mathcal{H}]$, which is a unital Banach algebra if and only if $\mathcal{H}$ is infinite dimensional (in a finite-dimensional space all operators are compact), and since spectra are nonempty in a unital Banach algebra, it follows that

$$\dim \mathcal{H} = \infty \quad \Longrightarrow \quad \sigma_e(T) \neq \varnothing.$$

However, if we take the expression in Corollary 5.12 as the definition of the essential spectrum, viz., $\sigma_e(T) = \{\lambda \in \mathbb{C} \colon (\lambda I - T) \in \mathcal{B}[\mathcal{H}] \backslash \mathcal{F}\}$, then the converse holds by Remark 5.3(c):

$$\dim \mathcal{H} < \infty \quad \Longrightarrow \quad \sigma_e(T) = \varnothing.$$

(b) ATKINSON THEOREM. An argument similar to that in the proof of Corollary 5.13, using each expression for $\sigma_{\ell e}(T)$ and $\sigma_{re}(T)$ in Corollary 5.11 separately, leads to the following result.

An operator $T \in \mathcal{B}[\mathcal{H}]$ lies in $\mathcal{F}_\ell$ (or in $\mathcal{F}_r$) if and only if its image $\pi(T)$ in the Calkin algebra $\mathcal{B}[\mathcal{H}]/\mathcal{B}_\infty[\mathcal{H}]$ is left (right) invertible.

(c) COMPACT PERTURBATION. Observe that, for every $K \in \mathcal{B}_\infty[\mathcal{H}]$,

$$\sigma_e(T + K) = \sigma_e(T).$$

Indeed, since $\pi(T + K) = \pi(T)$, it follows by the very definition of $\sigma_e(T)$ in $\mathcal{B}_\infty[\mathcal{H}]/\mathcal{B}[\mathcal{H}]$ that $\sigma_e(T) = \sigma(\pi(T)) = \sigma(\pi(T + K)) = \sigma_e(T + K)$, for every $K \in \mathcal{B}_\infty[\mathcal{H}]$. Similarly, the definitions of $\sigma_{\ell e}(T)$ and of $\sigma_{re}(T)$ in $\mathcal{B}_\infty[\mathcal{H}]/\mathcal{B}[\mathcal{H}]$ ensure that $\sigma_{\ell e}(T) = \sigma_\ell(\pi(T)) = \sigma_\ell(\pi(T + K)) = \sigma_{\ell e}(T + K)$, and $\sigma_{re}(T) = \sigma_r(\pi(T)) = \sigma_r(\pi(T + K)) = \sigma_{re}(T + K)$, for every $K \in \mathcal{B}_\infty[\mathcal{H}]$. Thus

$$\sigma_{\ell e}(T + K) = \sigma_{\ell e}(T) \quad \text{and} \quad \sigma_{re}(T + K) = \sigma_{re}(T).$$

This invariance consists of an important feature of Fredholm theory.

Now take any operator T in $\mathcal{B}[\mathcal{H}]$. For each $k \in \overline{\mathbb{Z}} \backslash \{0\}$ set

$$\sigma_k(T) = \big\{\lambda \in \mathbb{C} \colon \lambda I - T \in \mathcal{SF} \text{ and } \operatorname{ind}(\lambda I - T) = k\big\}.$$

Recall that a *component* of a set in a topological space is any maximal connected subset of it. A *hole* of a set in a topological space is any bounded component of its complement. If a set has an open complement (i.e., if it is closed), then a hole of it must be open. We shall show that each $\sigma_k(T)$ with finite index is a hole of $\sigma_e(T)$ that lies in $\sigma(T)$ (i.e., if $k \in \mathbb{Z} \backslash \{0\}$, then $\sigma_k(T) \subseteq \sigma(T) \backslash \sigma_e(T)$ is a bounded component of the open set $\mathbb{C} \backslash \sigma_e(T)$), and hence it is an open set. Moreover, we also show that $\sigma_{+\infty}(T)$ and $\sigma_{-\infty}(T)$ are holes of $\sigma_{re}(T)$ and of $\sigma_{\ell e}(T)$ that lie in $\sigma_{\ell e}(T)$ and $\sigma_{re}(T)$ (in fact, $\sigma_{+\infty}(T) = \sigma_{\ell e}(T) \backslash \sigma_{re}(T)$ is a bounded component of the open set $\mathbb{C} \backslash \sigma_{re}(T)$, and $\sigma_{-\infty}(T) = \sigma_{re}(T) \backslash \sigma_{\ell e}(T)$ is a bounded component of the open set $\mathbb{C} \backslash \sigma_{\ell e}(T)$), and hence they are open sets. The sets $\sigma_{+\infty}(T)$ and $\sigma_{-\infty}(T)$ — which are holes of $\sigma_{re}(T)$ and of $\sigma_{\ell e}(T)$ but are not holes of $\sigma_e(T)$ — are called *pseudoholes* of $\sigma_e(T)$.

Let $\sigma_{PF}(T)$ denote the set of all eigenvalues of T of finite multiplicity,

$$\sigma_{PF}(T) = \{\lambda \in \sigma_P(T) \colon \dim \mathcal{N}(\lambda I - T) < \infty\}$$
$$= \{\lambda \in \mathbb{C} \colon 0 < \dim \mathcal{N}(\lambda I - T) < \infty\}.$$

Observe from the diagram of Section 2.2 and Corollary 5.14 that

$$\sigma_{AP}(T) = \sigma_{\ell e}(T) \cup \sigma_{PF}(T).$$

Theorem 5.16. *For each* $k \in \overline{\mathbb{Z}} \backslash \{0\}$,

(a) $$\sigma_k(T) = \sigma_{-k}(T^*)^*.$$

If $k \in \mathbb{Z} \backslash \{0\}$, *then*

(b) $$\sigma_k(T) = \{\lambda \in \mathbb{C} \colon \lambda I - T \in \mathcal{F} \text{ and } \operatorname{ind}(\lambda I - T) = k\}$$
$$\subseteq \begin{cases} \sigma_{PF}(T), & 0 < k, \\ \sigma_{PF}(T^*)^*, & k < 0, \end{cases}$$

   so that $\sigma_k(T) \subseteq \sigma(T)$, *and*

$$\bigcup_{k \in \mathbb{Z} \backslash \{0\}} \sigma_k(T) = \{\lambda \in \mathbb{C} \colon \lambda I - T \in \mathcal{F} \text{ and } \operatorname{ind}(\lambda I - T) \neq 0\}$$
$$= \{\lambda \in \mathbb{C} \colon \lambda I - T \in \mathcal{F} \backslash \mathcal{W}\}$$
$$\subseteq \{\lambda \in \mathbb{C} \colon \lambda I - T \in \mathcal{F}\} = \mathbb{C} \backslash \sigma_e(T).$$

If $k = \pm\infty$, *then*

(c) $$\sigma_{+\infty}(T) \subseteq \sigma_P(T) \backslash \sigma_{PF}(T) \quad \text{and} \quad \sigma_{-\infty}(T) \subseteq \sigma_P(T^*)^* \backslash \sigma_{PF}(T^*)^*,$$

   and hence
$$\bigcup_{k \in \overline{\mathbb{Z}} \backslash \{0\}} \sigma_k(T) \subseteq \sigma_P(T) \cup \sigma_R(T) \subseteq \sigma(T).$$

   Moreover,

$$\sigma_{+\infty}(T) = \{\lambda \in \mathbb{C} \colon \lambda I - T \in \mathcal{SF} \text{ and } \operatorname{ind}(\lambda I - T) = +\infty\}$$
$$= \{\lambda \in \mathbb{C} \colon \lambda I - T \in \mathcal{F}_r \backslash \mathcal{F}_\ell\}$$
$$= \sigma_{\ell e}(T) \backslash \sigma_{re}(T) \subseteq \mathbb{C} \backslash \sigma_{re}(T),$$

$$\sigma_{-\infty}(T) = \{\lambda \in \mathbb{C} \colon \lambda I - T \in \mathcal{SF} \text{ and } \operatorname{ind}(\lambda I - T) = -\infty\}$$
$$= \{\lambda \in \mathbb{C} \colon \lambda I - T \in \mathcal{F}_\ell \backslash \mathcal{F}_r\}$$
$$= \sigma_{re}(T) \backslash \sigma_{\ell e}(T) \subseteq \mathbb{C} \backslash \sigma_{\ell e}(T),$$

   which are all open subsets of $\mathbb{C}$, *and hence*

$$\sigma_{+\infty}(T) \subseteq \sigma_e(T) \quad \text{and} \quad \sigma_{-\infty}(T) \subseteq \sigma_e(T).$$

Furthermore, if

(d) $\mathbb{Z}^{+}\backslash\{0\} = \overline{\mathbb{N}}$ *denotes the set of all extended positive integers, and* $\mathbb{Z}^{-}\backslash\{0\} = -\overline{\mathbb{N}}$ *denotes the set of all extended negative integers, then*

$$\sigma_{P_1}(T) \subseteq \bigcup_{k \in \overline{\mathbb{Z}}^{+}\backslash\{0\}} \sigma_k(T)$$
$$= \{\lambda \in \mathbb{C}: \lambda I - T \in \mathcal{SF} \ \ and \ \ 0 < \mathrm{ind}\,(\lambda I - T)\},$$

$$\sigma_{R_1}(T) \subseteq \bigcup_{k \in \overline{\mathbb{Z}}^{-}\backslash\{0\}} \sigma_k(T)$$
$$= \{\lambda \in \mathbb{C}: \lambda I - T \in \mathcal{SF} \ \ and \ \ \mathrm{ind}\,(\lambda I - T) < 0\},$$

so that

$$\sigma_{P_1}(T) \cup \sigma_{R_1}(T) \subseteq \bigcup_{k \in \overline{\mathbb{Z}}\backslash\{0\}} \sigma_k(T)$$
$$= \{\lambda \in \mathbb{C}: \lambda I - T \in \mathcal{SF} \ \ and \ \ \mathrm{ind}\,(\lambda I - T) \neq 0\}$$
$$\subseteq \{\lambda \in \mathbb{C}: \lambda I - T \in \mathcal{SF}\} = \mathbb{C}\backslash\big(\sigma_{\ell e}(T) \cap \sigma_{re}(T)\big),$$

which are all open subsets of $\mathbb{C}$.

(e) *If k is finite, then $\sigma_k(T)$ is a hole of $\sigma_{\ell e}(T) \cap \sigma_{re}(T)$ (and therefore, a hole of $\sigma_e(T)$). Otherwise, $\sigma_{+\infty}(T)$ is a hole of $\sigma_{\ell e}(T)$ and $\sigma_{-\infty}(T)$ is a hole of $\sigma_{re}(T)$, and so $\sigma_k(T)$ is an open subset of $\mathbb{C}$ for every $k \in \overline{\mathbb{Z}}\backslash\{0\}$.*

Proof. (a) Recall that $\lambda I - T \in \mathcal{SF}$ if and only if $\overline{\lambda} I - T^* \in \mathcal{SF}$, and also that $\mathrm{ind}\,(\lambda I - T) = -\mathrm{ind}\,(\overline{\lambda} I - T^*)$. Hence $\lambda \in \sigma_k(T)$ if and only if $\overline{\lambda} \in \sigma_{-k}(T^*)$.

(b) The expression for $\sigma_k(T)$ — with $\mathcal{SF}$ replaced with $\mathcal{F}$ — holds since for finite k all semi-Fredholm operators are Fredholm (cf. Remark 5.3(e)). Take any $k \in \mathbb{Z}\backslash\{0\}$. If $k > 0$ and $\lambda \in \sigma_k(T)$, then $0 < \mathrm{ind}\,(\lambda I - T) < \infty$. Hence

$$0 \leq \dim \mathcal{N}(\overline{\lambda} I - T^*) < \dim \mathcal{N}(\lambda I - T) < \infty,$$

and so $0 < \dim \mathcal{N}(\lambda I - T) < \infty$. Thus λ is an eigenvalue of T of finite multiplicity. Dually, if $k < 0$ (i.e., $-k > 0$) and $\lambda \in \sigma_k(T) = \sigma_{-k}(T^*)^*$, then $\overline{\lambda}$ is an eigenvalue of T^* of finite multiplicity. To close the proof of (b), observe that

$$\mathbb{C}\backslash\sigma_e(T) = \{\lambda \in \mathbb{C}: \lambda I - T \in \mathcal{F}\}$$

according to Corollary 5.12.

(c) If $\lambda \in \sigma_{+\infty}(T)$, then

$$0 \leq \dim \mathcal{N}(\overline{\lambda} I - T^*) < \dim \mathcal{N}(\lambda I - T) = \infty,$$

so that λ is an eigenvalue of T of infinite multiplicity. Dually, if $\lambda \in \sigma_{-\infty}(T) = \sigma_{+\infty}(T^*)^*$, then $\overline{\lambda}$ is an eigenvalue of T^* of infinite multiplicity. The expressions for $\sigma_{+\infty}(T)$ and $\sigma_{-\infty}(T)$ follow from Remark 5.3(e) and also from Corollary 5.11. Recall that $\sigma_{\ell e}(T)$ and $\sigma_{re}(T)$ are closed in $\mathbb{C}$, and so $\sigma_{+\infty}(T)$ and

$\sigma_{-\infty}(T)$ are open in $\mathbb{C}$. Moreover, the expressions for $\sigma_{+\infty}(T)$ and $\sigma_{-\infty}(T)$ and Corollary 5.12 ensure that these sets are subsets of $\sigma_e(T)$.

(d) Recall from Section 1.3 that $\mathcal{M}^- = \mathcal{H}$ if and only if $\mathcal{M}^\perp = \{0\}$. Hence $\mathcal{R}(\lambda I - T)^- = \mathcal{H}$ if and only if $\mathcal{N}(\overline{\lambda} I - T^*) = \{0\}$ by Lemma 1.4. In other words, $\mathcal{R}(\lambda I - T)^- = \mathcal{H}$ if and only if $\dim \mathcal{N}(\overline{\lambda} I - T^*) = 0$. Thus, according to the diagram of Section 2.2, we get

$$\sigma_{P_1}(T) = \big\{\lambda \in \mathbb{C} \colon \mathcal{R}(\lambda I - T) \text{ is closed}, \ 0 = \dim \mathcal{N}(\overline{\lambda} I - T^*) \neq \dim \mathcal{N}(\lambda I - T)\big\}$$
$$= \big\{\lambda \in \mathbb{C} \colon \mathcal{R}(\lambda I - T) \text{ is closed}, \ 0 = \dim \mathcal{N}(\overline{\lambda} I - T^*), \ \mathrm{ind}\,(\lambda I - T) > 0\big\}.$$

Therefore, by Corollary 5.2,

$$\sigma_{P_1}(T) = \big\{\lambda \in \mathbb{C} \colon \lambda I - T \in \mathcal{SF}, \ 0 = \dim \mathcal{N}(\lambda I - T), \ \mathrm{ind}\,(\lambda I - T) > 0\big\}$$
$$\subseteq \big\{\lambda \in \mathbb{C} \colon \lambda I - T \in \mathcal{SF}, \ \mathrm{ind}\,(\lambda I - T) > 0\big\} = \bigcup\nolimits_{k \in \overline{\mathbb{Z}}^+ \setminus \{0\}} \sigma_k(T).$$

Dually (cf. item (a) and Theorem 2.6),

$$\sigma_{R_1}(T) = \sigma_{P_1}(T^*)^* \subseteq \Big(\bigcup\nolimits_{k \in \overline{\mathbb{Z}}^+ \setminus \{0\}} \sigma_k(T^*)\Big)^*$$
$$= \Big(\bigcup\nolimits_{k \in \overline{\mathbb{Z}}^+ \setminus \{0\}} \sigma_{-k}(T)^*\Big)^* = \bigcup\nolimits_{k \in \overline{\mathbb{Z}}^- \setminus \{0\}} \sigma_k(T).$$

Finally, note that

$$\mathbb{C} \setminus \sigma_{\ell e}(T) \cap \sigma_{re}(T) = \big\{\lambda \in \mathbb{C} \colon \lambda I - T \in \mathcal{SF}\big\}$$

according to Corollary 5.11, as we had verified before.

(e) Recall from item (d) that

$$\bigcup\nolimits_{k \in \overline{\mathbb{Z}} \setminus \{0\}} \sigma_k(T) \subseteq \mathbb{C} \setminus \sigma_{\ell e}(T) \cap \sigma_{re}(T) = \big\{\lambda \in \mathbb{C} \colon \lambda I - T \in \mathcal{SF}\big\}.$$

Take any $\lambda \in \bigcup_{k \in \overline{\mathbb{Z}} \setminus \{0\}} \sigma_k(T)$ so that $\lambda I - T \in \mathcal{SF}$. Proposition 5.D ensures that $\mathrm{ind}\,(\nu I - T)$ is constant for every ν in a punctured disk $B_\varepsilon(\lambda) \setminus \{0\}$ centered at λ for some positive radius ε. Thus, for every pair of distinct extended integers $j, k \in \overline{\mathbb{Z}} \setminus \{0\}$, the sets $\sigma_j(T)$ and $\sigma_k(T)$ are not only disjoint, but they are (bounded) components (i.e., maximal connected subsets) of the open set $\mathbb{C} \setminus \sigma_{\ell e}(T) \cap \sigma_{re}(T)$, and hence they are holes of the closed set $\sigma_{\ell e}(T) \cap \sigma_{re}(T)$. By item (b), if j, k are finite (i.e., if $j, k \in \mathbb{Z} \setminus \{0\}$), then $\sigma_j(T)$ and $\sigma_k(T)$ are (bounded) components of the open set $\mathbb{C} \setminus \sigma_e(T) = \mathbb{C} \setminus \sigma_{\ell e}(T) \cup \sigma_{re}(T)$ and, in this case, they are holes of the closed set $\sigma_e(T)$. By item (c), $\sigma_{+\infty}(T)$ and $\sigma_{-\infty}(T)$ are (bounded) components of the open sets $\mathbb{C} \setminus \sigma_{re}(T)$ and $\mathbb{C} \setminus \sigma_{\ell e}(T)$, and so $\sigma_{+\infty}(T)$ and $\sigma_{-\infty}(T)$ are holes of the closed sets $\sigma_{re}(T)$ and $\sigma_{\ell e}(T)$, respectively. Thus, in any case, $\sigma_k(T)$ is an open subset of $\mathbb{C}$. Therefore, $\bigcup_{k \in \overline{\mathbb{Z}} \setminus \{0\}} \sigma_k(T)$ is open as claimed in (d). $\qquad\square$

The following figure illustrates a simple instance of holes and pseudoholes of the essential spectrum. Observe that the right and left essential spectra, $\sigma_{\ell e}(T)$ and $\sigma_{re}(T)$, differ only by pseudoholes $\sigma_{\pm\infty}(T)$.
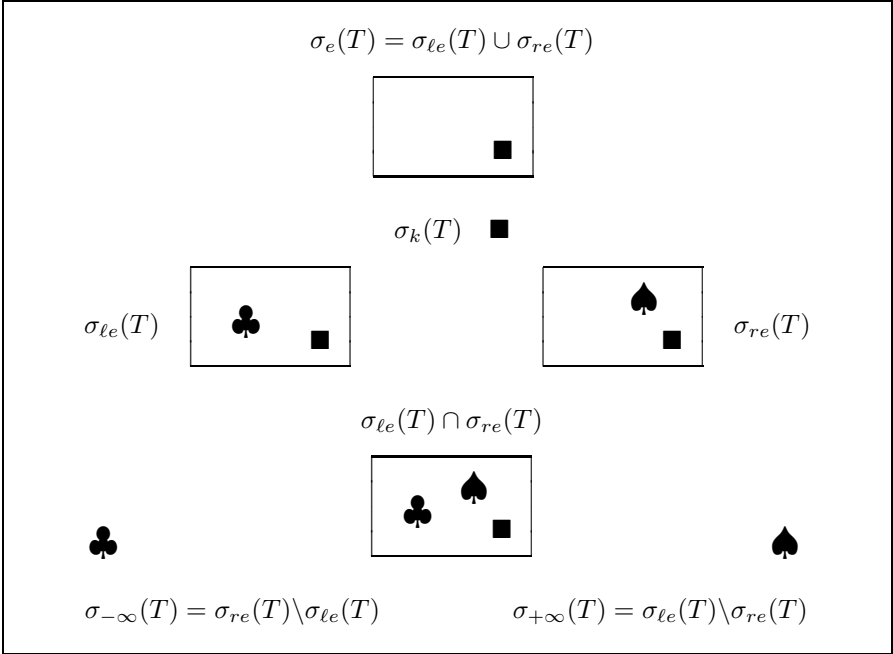
Fig. § 5.2. Holes and pseudoholes of the essential spectrum

Now, for $k = 0$, we define $\sigma_0(T)$ as the following subset of $\sigma(T)$.

$$\sigma_0(T) = \big\{\lambda \in \sigma(T)\colon \lambda I - T \in \mathcal{SF} \text{ and } \operatorname{ind}(\lambda I - T) = 0\big\}$$
$$= \big\{\lambda \in \sigma(T)\colon \lambda I - T \in \mathcal{F} \text{ and } \operatorname{ind}(\lambda I - T) = 0\big\}$$
$$= \big\{\lambda \in \sigma(T)\colon \lambda I - T \in \mathcal{W}\big\}.$$

According to Corollary 5.2, the diagram of Section 2.2, and Theorem 2.6, such a subset of the spectrum can be rewritten as

$$\sigma_0(T) = \big\{\lambda \in \sigma(T)\colon \mathcal{R}(\lambda I - T) \text{ is closed and}$$
$$\dim \mathcal{N}(\lambda I - T) = \dim \mathcal{N}(\overline{\lambda} I - T^*) < \infty\big\}$$
$$= \big\{\lambda \in \sigma_P(T)\colon \mathcal{R}(\lambda I - T) = \mathcal{R}(\lambda I - T)^- \neq \mathcal{H} \text{ and}$$
$$\dim \mathcal{N}(\lambda I - T) = \dim \mathcal{N}(\overline{\lambda} I - T^*) < \infty\big\}$$
$$= \big\{\lambda \in \sigma_{P_4}\colon \dim \mathcal{N}(\lambda I - T) = \dim \mathcal{N}(\overline{\lambda} I - T^*) < \infty\big\},$$

where $\sigma_{P_4}(T) = \big\{\lambda \in \sigma_P(T)\colon \mathcal{R}(\lambda I - T)^- = \mathcal{R}(\lambda I - T) \neq \mathcal{H}\big\}$. In fact,

$$\sigma_0(T) \subseteq \sigma_{PF}(T) \cap \sigma_{PF}(T^*)^* \subseteq \sigma_P(T) \cap \sigma_P(T^*)^*.$$

Thus

$$\sigma_0(T) = \sigma_0(T^*)^*.$$

Remark 5.17. (a) FREDHOLM ALTERNATIVE. According to Remark 5.3(d), if $K \in \mathcal{B}_\infty[\mathcal{H}]$ and $\lambda \neq 0$, then $\lambda I - K$ is Fredholm with $\operatorname{ind}(\lambda I - K) = 0$, which means that $\lambda I - K$ is Weyl (i.e., $\lambda I - K \in \mathcal{W}$ — see Remark 5.7(b)). Since $\sigma_0(K) = \{\lambda \in \sigma(K) : \lambda I - K \in \mathcal{W}\}$, the Fredholm Alternative can be restated, once again, as follows (compare with Theorem 2.18):

$$K \in \mathcal{B}_\infty[\mathcal{H}] \quad \Longrightarrow \quad \sigma(K) \backslash \{0\} = \sigma_0(K) \backslash \{0\}.$$

(b) FINITE DIMENSIONAL. Remark 5.3(c) says that if $\mathcal{H}$ is finite dimensional, then every operator is Weyl (i.e., if $\dim \mathcal{H} < \infty$, then $\mathcal{W} = \mathcal{B}[\mathcal{H}]$). Thus,

$$\dim \mathcal{H} < \infty \quad \Longrightarrow \quad \sigma(T) = \sigma_0(T).$$

(c) COMPACT PERTURBATION. The expressions for $\sigma_k(T), \sigma_{+\infty}(T)$, and $\sigma_{+\infty}(T)$, viz., $\sigma_k(T) = \{\lambda \in \mathbb{C} : \lambda I - T \in \mathcal{F}$ and $\operatorname{ind}(\lambda I - T) = k\}$ for every $k \in \mathbb{Z} \backslash \{0\}$, $\sigma_{+\infty}(T) = \sigma_{\ell e}(T) \backslash \sigma_{re}(T)$, and $\sigma_{-\infty}(T) = \sigma_{re}(T) \backslash \sigma_{\ell e}(T)$ as in Theorem 5.16, together with the results of Theorem 5.4, ensure that the sets $\sigma_k(T)$ for each $k \in \mathbb{Z} \backslash \{0\}$, $\sigma_{+\infty}(T)$, and $\sigma_{+\infty}(T)$, also are invariant under compact perturbation (Remark 5.15(c)). That is, for every $K \in \mathcal{B}_\infty[\mathcal{H}]$,

$$\sigma_k(T + K) = \sigma_k(T) \quad \text{for every} \quad k \in \overline{\mathbb{Z}} \backslash \{0\}.$$

However, such an invariance does not apply to $\sigma_0(T)$. Why? Observe that if $\lambda \in \sigma_0(T)$, then the definition of $\sigma_0(T)$ forces λ to be in $\sigma(T)$, while in the definition of $\sigma_k(T)$ for $k \neq 0$ this happens naturally. In fact, the definition of $\sigma_0(T)$ forces λ to be in $\sigma_P(T)$. Example: If $\dim \mathcal{H} < \infty$ (where all operators are compact), and if $T = I$, then we get $\sigma_0(T) = \sigma(T) = \{1\}$ and $\sigma_0(T - T) = \sigma(T - T) = \{0\}$. That is, there is a compact $K = -T$ such that

$$\{0\} = \sigma_0(T + K) \neq \sigma_0(T) = \{1\}.$$

This will be generalized in the proof of Theorem 5.24, where it is shown that $\sigma_0(T)$ *is never invariant under compact perturbation.*

The partition of the spectrum $\sigma(T)$ obtained in the following corollary is called the *spectral picture* of T [75].

Corollary 5.18. (SPECTRAL PICTURE). *If* $T \in \mathcal{B}[\mathcal{H}]$, *then*

$$\sigma(T) = \sigma_e(T) \cup \bigcup_{k \in \mathbb{Z}} \sigma_k(T)$$

and

$$\sigma_e(T) = \big(\sigma_{\ell e}(T) \cap \sigma_{re}(T)\big) \cup \sigma_{+\infty}(T) \cup \sigma_{-\infty}(T),$$

where

$$\sigma_e(T) \cap \bigcup_{k \in \mathbb{Z}} \sigma_k(T) = \varnothing,$$

$$\sigma_k(T) \cap \sigma_j(T) = \varnothing \quad \textit{for every} \quad j, k \in \overline{\mathbb{Z}} \textit{ such that } j \neq k,$$

and

$$\big(\sigma_{\ell e}(T) \cap \sigma_{re}(T)\big) \cap \big(\sigma_{+\infty}(T) \cup \sigma_{+\infty}\big) = \varnothing.$$

Proof. The collection $\{\sigma_k(T)\}_{k \in \overline{\mathbb{Z}}}$ is clearly pairwise disjoint. According to Theorem 5.16(b), and by the definition of $\sigma_0(T)$,

$$\bigcup_{k \in \mathbb{Z}} \sigma_k(T) = \big\{\lambda \in \sigma(T) \colon \lambda I - T \in \mathcal{F}\big\}.$$

Since $\sigma_e(T) \subseteq \sigma(T)$, it follows by Corollary 5.12 that

$$\sigma_e(T) = \big\{\lambda \in \sigma(T) \colon \lambda I - T \notin \mathcal{F}\big\}.$$

Hence we get the partition of the spectrum: $\sigma(T) = \sigma_e(T) \cup \bigcup_{k \in \mathbb{Z}} \sigma_k(T)$ with $\sigma_e(T) \cap \bigcup_{k \in \mathbb{Z}} \sigma_k(T) = \varnothing$. Since $\sigma_e(T) = \sigma_{\ell e}(T) \cup \sigma_{re}(T) \subseteq \sigma(T)$, it also follows from Corollary 5.11 that

$$\sigma_{\ell e}(T) \cap \sigma_{re}(T) = \big\{\lambda \in \sigma(T) \colon \lambda I - T \notin \mathcal{SF}\big\} \subseteq \sigma_e(T).$$

Moreover, Theorem 5.16(c) ensures that

$$\sigma_{+\infty}(T) \cup \sigma_{-\infty}(T) = \big\{\lambda \in \sigma(T) \colon \lambda I - T \in \mathcal{SF} \backslash \mathcal{F}\big\},$$

and that $\sigma_{+\infty}(T) \cap \sigma_{-\infty}(T) = \varnothing$. Thus we get the following partition of the essential spectrum: $\sigma_e(T) = \big(\sigma_{\ell e}(T) \cap \sigma_{re}(T)\big) \cup \sigma_{+\infty}(T) \cup \sigma_{-\infty}(T)$ with $\big(\sigma_{\ell e}(T) \cap \sigma_{re}(T)\big) \cap \big(\sigma_{+\infty}(T) \cup \sigma_{+\infty}\big) = \varnothing.$ $\square$

Recall that the collection $\{\sigma_k(T)\}_{k \in \mathbb{Z} \backslash \{0\}}$ consists of pairwise disjoint *open* sets (cf. Theorem 5.16), which are subsets of $\sigma(T) \backslash \sigma_e(T)$. These are the *holes* of the essential spectrum $\sigma_e(T)$. Moreover, $\sigma_{\pm\infty}(T)$ also are *open* sets (cf. Theorem 5.16), which are subsets of $\sigma_e(T)$. These are the *pseudoholes* of $\sigma_e(T)$ (they are holes of $\sigma_{re}(T)$ and $\sigma_{\ell e}(T)$). Thus the spectral picture of T consists of the essential spectrum $\sigma_e(T)$, the holes $\sigma_k(T)$ and pseudoholes $\sigma_{\pm\infty}(T)$ (to each is associated a nonzero index k in $\overline{\mathbb{Z}} \backslash \{0\}$), and the set $\sigma_0(T)$. It is worth noticing that any spectral picture can be attained [25] by an operator in $\mathcal{B}[\mathcal{H}]$ (see also Proposition 5.G).

5.3 Riesz Points and Weyl Spectrum

The set $\sigma_0(T)$ will play a rather important role in the sequel. It consists of an open set $\tau_0(T)$ and the set $\pi_0(T)$ of isolated points of $\sigma(T)$ for which their Riesz idempotents have finite rank. The next proposition says that $\pi_0(T)$ is

precisely the set of all isolated points of $\sigma(T)$ that lie in $\sigma_0(T)$. Let $E_\lambda \in \mathcal{B}[\mathcal{H}]$ be the Riesz idempotent associated with an isolated point λ of the spectrum $\sigma(T)$ of an operator $T \in \mathcal{B}[\mathcal{H}]$.

Theorem 5.19. *If λ is an isolated point of $\sigma(T)$, then the following assertions are pairwise equivalent.*

(a) $\lambda \in \sigma_0(T)$.

(b) $\lambda \notin \sigma_e(T)$.

(c) $\lambda I - T \in \mathcal{F}$.

(d) $\mathcal{R}(\lambda I - T)$ *is closed and* $\dim \mathcal{N}(\lambda I - T) < \infty$.

(e) $\lambda \notin \sigma_{\ell e}(T) \cap \sigma_{re}(T)$.

(f) $\dim \mathcal{R}(E_\lambda) < \infty$.

Proof. Let T be an operator on an infinite-dimensional complex Hilbert space.

(a)$\Rightarrow$(b)$\Rightarrow$(c). By definition, we have $\sigma_0(T) = \{\lambda \in \sigma(T) \colon \lambda I - T \in \mathcal{W}\} = \{\lambda \in \sigma(T) \colon \lambda I - T \in \mathcal{F}$ and $\mathrm{ind}\,(\lambda I - T) = 0\}$. Thus (a) implies (c) tautologically, and (c) is equivalent to (b) by Corollary 5.12.

(c)$\Rightarrow$(d). By Corollary 5.2, $\lambda I - T \in \mathcal{F}$ if and only if $\mathcal{R}(\lambda I - T)$ is closed, $\dim \mathcal{N}(\lambda I - T) < \infty$, and $\dim \mathcal{N}(\overline{\lambda} I - T^*) < \infty$. Thus (c) implies (d) trivially.

(d)$\Rightarrow$(e). By Corollary 5.14, $\sigma_{\ell e}(T) \cap \sigma_{re}(T) = \{\lambda \in \mathbb{C} \colon \mathcal{R}(\lambda I - T)$ is not closed or $\dim \mathcal{N}(\lambda I - T) = \dim \mathcal{N}(\overline{\lambda} I - T^*) = \infty\}$. Thus (d) implies (e).

From now on suppose λ is an isolated point of the spectrum $\sigma(T)$ of T.

(e)$\Rightarrow$(a). Observe that, if $\lambda \in \sigma(T) \backslash \sigma_0(T)$, then $\lambda \in \sigma_e(T) \cup \bigcup_{k \in \mathbb{Z} \backslash \{0\}} \sigma_k(T)$ by Corollary 5.18, where $\bigcup_{k \in \mathbb{Z} \backslash \{0\}} \sigma_k(T)$ is an open subset of $\mathbb{C}$ by Theorem 5.16. So it has no isolated point. Thus, if λ is an isolated pont of $\sigma(T) \backslash \sigma_0(T)$, then λ lies in $\sigma_e(T) = (\sigma_{\ell e}(T) \cap \sigma_{re}(T)) \cup (\sigma_{+\infty}(T) \cup \sigma_{-\infty}(T))$, where the set $\sigma_{+\infty}(T) \cup \sigma_{-\infty}(T)$ is open in $\mathbb{C}$, and hence it has no isolated point as well (cf. Theorem 5.16 and Corollary 5.18 again). Therefore, $\lambda \in \sigma_{\ell e}(T) \cap \sigma_{re}(T)$. In other words, if λ is an isolated point of $\sigma(T)$ and $\lambda \notin \sigma_{\ell e}(T) \cap \sigma_{re}(T)$, then $\lambda \in \sigma_0(T)$. Thus (e) implies (a).

This concludes the proof that (a), (b), (c), (d), and (e) are pairwise equivalent. Next we show that (f) also is equivalent to them.

(f)$\Rightarrow$(d). If λ is an isolated point of $\sigma(T)$, then $\Delta = \{\lambda\}$ is a spectral set for the operator T. Thus Corollary 4.22 ensures that (f) implies (d).

(a)$\Rightarrow$(f). Let $\pi \colon \mathcal{B}[\mathcal{H}] \to \mathcal{B}[\mathcal{H}]/\mathcal{B}_\infty[\mathcal{H}]$ be the natural map of $\mathcal{B}[\mathcal{H}]$ into the Calkin algebra $\mathcal{B}[\mathcal{H}]/\mathcal{B}_\infty[\mathcal{H}]$, which is a unital homomorphism of the unital Banach algebra $\mathcal{B}[\mathcal{H}]$ to the unital Banach algebra $\mathcal{B}[\mathcal{H}]/\mathcal{B}_\infty[\mathcal{H}]$, whenever the complex Hilbert space $\mathcal{H}$ is infinite dimensional.

Claim. If $T \in \mathcal{B}[\mathcal{H}]$ and $\nu \in \rho(T)$, then $\nu \in \rho(\pi(T))$ and

$$\pi\big((\nu I - T)^{-1}\big) = \big(\nu\pi(I) - \pi(T)\big)^{-1}.$$

Proof. Observe that, since $\mathcal{B}_\infty[\mathcal{H}]$ is an ideal of $\mathcal{B}[\mathcal{H}]$,

$$\begin{aligned}
\pi\big((\nu I - T)^{-1}\big)\,\pi(\nu I - T) &= \big((\nu I - T)^{-1} + \mathcal{B}_\infty[\mathcal{H}]\big)\big((\nu I - T) + \mathcal{B}_\infty[\mathcal{H}]\big) \\
&= (\nu I - T)^{-1}(\nu I - T) + \mathcal{B}_\infty[\mathcal{H}] \\
&= I + \mathcal{B}_\infty[\mathcal{H}] = \pi(I) = [I],
\end{aligned}$$

where $\pi(I) = [I]$ is the identity in $\mathcal{B}[\mathcal{H}]/\mathcal{B}_\infty[\mathcal{H}]$. Therefore,

$$\pi\big((\nu I - T)^{-1}\big) = \big(\pi(\nu I - T)\big)^{-1} = \big(\nu I - T + \mathcal{B}_\infty[\mathcal{H}]\big)^{-1} = \big(\nu\pi(I) - \pi(T)\big)^{-1},$$

completing the proof of the claimed result.

Suppose the isolated point λ of $\sigma(T)$ lies in $\sigma_0(T)$. According to Corollary 5.18, $\lambda \notin \sigma_e(T)$, which means, by definition, that $\lambda \notin \sigma(\pi(T))$, and hence $\lambda \in \rho(\pi(T))$. That is, $\lambda\pi(I) - \pi(T)$ is invertible in $\mathcal{B}[\mathcal{H}]/\mathcal{B}_\infty[\mathcal{H}]$. Since the resolvent function of an element in a unital complex Banach algebra is analytic on the resolvent set (cf. proof of Claim 1 in the proof of Lemma 4.13), it then follows that $(\nu\pi(I) - \pi(T))^{-1}\colon \Lambda \to \mathcal{B}[\mathcal{H}]/\mathcal{B}_\infty[\mathcal{H}]$ is analytic on any nonempty open subset Λ of $\mathbb{C}$ such that $(\mathrm{ins}\,\Gamma_\lambda)^- \subset \Lambda$, where Γ_λ is any simple closed rectifiable positively oriented curve (e.g., any positively oriented circle) enclosing λ but no other point of $\sigma(T)$. Thus the Cauchy Theorem (cf. Claim 2 in the proof of Lemma 4.13) says that $\int_{\Gamma_\lambda}(\nu\pi(I) - \pi(T))^{-1}d\nu = [O]$, where $[O] = \pi(O) = \mathcal{B}_\infty[\mathcal{H}]$ is the origin in Calkin algebra $\mathcal{B}[\mathcal{H}]/\mathcal{B}_\infty[\mathcal{H}]$. Then, with $E_\lambda = \frac{1}{2\pi i}\int_{\Gamma_\lambda}(\nu I - T)^{-1}\,d\nu$ standing for the Riesz idempotent associated with λ, and since $\pi\colon \mathcal{B}[\mathcal{H}] \to \mathcal{B}[\mathcal{H}]/\mathcal{B}_\infty[\mathcal{H}]$ is linear and bounded,

$$\pi(E_\lambda) = \tfrac{1}{2\pi i}\int_{\Gamma_\lambda} \pi\big((\nu I - T)^{-1}\big)\,d\nu = \tfrac{1}{2\pi i}\int_{\Gamma_\lambda}\big(\nu\pi(I) - \pi(T)\big)^{-1}d\nu = [O],$$

which means that $E_\lambda \in \mathcal{B}_\infty[\mathcal{H}]$. But the restriction of a compact operator to an invariant subspace is again compact (cf. Proposition 1.V), and so the identity $I = E_\lambda|_{\mathcal{R}(E_\lambda)}\colon \mathcal{R}(E_\lambda) \to \mathcal{R}(E_\lambda)$ on $\mathcal{R}(E_\lambda)$ is compact, and therefore $\dim(R(E_\lambda)) < \infty$ (cf. Proposition 1.Y). Thus (a) implies (f). $\qquad\square$

A *Riesz point* of an operator T is an isolated point λ of $\sigma(T)$ for which the Riesz idempotent E_λ has finite rank (i.e., for which $\dim\mathcal{R}(E_\lambda) < \infty$). Let $\sigma_{\mathrm{iso}}(T)$ denote the set of all isolated points of the spectrum $\sigma(T)$,

$$\sigma_{\mathrm{iso}}(T) = \big\{\lambda \in \sigma(T)\colon \lambda \text{ is an isolated point of } \sigma(T)\big\},$$

so that its complement in $\sigma(T)$,

$$\sigma_{\mathrm{acc}}(T) = \sigma(T)\backslash\sigma_{\mathrm{iso}}(T),$$

is the set of all accumulation points of $\sigma(T)$. Let $\pi_0(T)$ denote the set of all isolated points of $\sigma(T)$ that lie in $\sigma_0(T)$; that is,

$$\pi_0(T) = \sigma_{\text{iso}}(T) \cap \sigma_0(T).$$

Recall: $\sigma_0(T) \subseteq \sigma_{P_4}(T) \subseteq \sigma_P(T)$. Thus Theorem 5.19 says that $\pi_0(T)$ is precisely *the set of all Riesz points of* T, which are eigenvalues of T,

$$\pi_0(T) = \big\{ \lambda \in \sigma_{\text{iso}}(T) \colon \dim \mathcal{R}(E_\lambda) < \infty \big\} \subseteq \sigma_P(T),$$

and also that

$$\sigma_{\text{iso}}(T) \backslash \sigma_e(T) = \sigma_{\text{iso}}(T) \cap \sigma_0(T) \subseteq \sigma_{P_4}(T) \subseteq \sigma_P(T).$$

Summing up (cf. Theorem 5.19 and the diagram of Section 2.2):

$$\pi_0(T) = \big\{ \lambda \in \sigma_P(T) \colon \lambda \in \sigma_{\text{iso}}(T) \text{ and } \lambda \in \sigma_0(T) \big\} = \sigma_{\text{iso}}(T) \cap \sigma_0(T)$$
$$= \big\{ \lambda \in \sigma_P(T) \colon \lambda \in \sigma_{\text{iso}}(T) \text{ and } \lambda \notin \sigma_e(T) \big\} = \sigma_{\text{iso}}(T) \backslash \sigma_e(T)$$
$$= \big\{ \lambda \in \sigma_P(T) \colon \lambda \in \sigma_{\text{iso}}(T) \text{ and } \dim \mathcal{R}(E_\lambda) < \infty \big\}$$
$$= \big\{ \lambda \in \sigma_P(T) \colon \lambda \in \sigma_{\text{iso}}(T) \text{ and } \lambda \notin \big(\sigma_{\ell e}(T) \cap \sigma_{re}(T) \big) \big\}$$
$$= \big\{ \lambda \in \sigma_P(T) \colon \lambda \in \sigma_{\text{iso}}(T), \ \dim \mathcal{N}(\lambda I - T) < \infty, \text{ and } \mathcal{R}(\lambda I - T) \text{ is closed} \big\}$$
$$= \big\{ \lambda \in \sigma_{P_4}(T) \colon \lambda \in \sigma_{\text{iso}}(T) \text{ and } \dim \mathcal{N}(\lambda I - T) = \dim \mathcal{N}(\overline{\lambda} I - T^*) < \infty \big\}.$$

Let $\tau_0(T)$ denote the complement of $\pi_0(T)$ in $\sigma_0(T)$,

$$\tau_0(T) = \sigma_0(T) \backslash \pi_0(T) \subseteq \sigma_{P_4}(T),$$

so that $\{\pi_0(T), \tau_0(T)\}$ is a partition of $\sigma_0(T)$. That is, $\sigma_0(T) \cap \pi(T) = \varnothing$ and

$$\sigma_0(T) = \tau_0(T) \cup \pi_0(T).$$

Corollary 5.20. $\tau_0(T)$ is an open subset of $\mathbb{C}$.

Proof. Observe that $\tau_0(T) = \sigma_0(T) \backslash \pi_0(T) \subseteq \sigma(T)$ has no isolated point. Moreover, if $\lambda \in \tau_0(T)$, then $\lambda I - T \in \mathcal{W}$ and $\lambda \in \mathbb{C} \backslash \sigma_{\ell e}(T) \cap \sigma_{re}(T)$ (by Theorem 5.19). Thus (since λ is not an isolated point), it follows by Proposition 5.D that there exists a nonempty open ball $B_\varepsilon(\lambda)$ centered at λ and included in $\tau_0(T)$. Therefore $\tau_0(T)$ is an open subset of $\mathbb{C}$. $\square$

Let $\partial \sigma(T)$ denote the boundary of $\sigma(T)$, as usual.

Corollary 5.21. *If* $\lambda \in \partial \sigma(T)$, *then either* λ *is an isolated point of* $\sigma(T)$ *in* $\pi_0(T)$ *or* $\lambda \in \sigma_{\ell e}(T) \cap \sigma_{re}(T)$.

Proof. If $\lambda \in \partial \sigma(T)$, then $\lambda \notin \big(\bigcup_{k \in \mathbb{Z} \backslash \{0\}} \sigma_k(T) \cup \sigma_{+\infty}(T) \cup \sigma_{-\infty}(T) \big)$ because this is a subset of the closed set $\sigma(T)$ that is open in $\mathbb{C}$ (cf. Theorem 5.16). Thus $\lambda \in (\sigma_{\ell e}(T) \cap \sigma_{re}(T)) \cup \sigma_0(T)$ by Corollary 5.18. But $\sigma_0(T) = \tau_0(T) \cup \pi_0(T)$,

where $\pi_0(T) = \sigma_{\mathrm{iso}}(T) \cap \sigma_0(T)$, and so $\pi_0(T)$ consists of isolated points only. However $\lambda \notin \tau_0(T)$ because $\tau_0(T)$ also is a subset of closed set $\sigma(T)$ that is open in $\mathbb{C}$ (Corollary 5.20). Thus $\lambda \in (\sigma_{\ell e}(T) \cap \sigma_{re}(T)) \cup \pi_0(T)$. $\qquad \square$

Let $\pi_{00}(T)$ denote the set of all isolated eigenvalues of finite multiplicity,

$$\pi_{00}(T) = \sigma_{\mathrm{iso}}(T) \cap \sigma_{PF}(T).$$

Since $\sigma_0(T) \subseteq \sigma_{PF}(T)$, it follows that $\pi_0(T) \subseteq \pi_{00}(T)$. Indeed,

$$\pi_0(T) = \sigma_{\mathrm{iso}}(T) \cap \sigma_0(T) \subseteq \sigma_{\mathrm{iso}}(T) \cap \sigma_{PF}(T) = \pi_{00}(T).$$

Corollary 5.22. $\pi_0(T) = \{\lambda \in \pi_{00}(T) : \mathcal{R}(\lambda I - T) \text{ is closed}\}.$

Proof. If $\lambda \in \pi_0(T)$, then $\lambda \in \pi_{00}(T)$ and $\mathcal{R}(\lambda I - T)$ is closed (since $\lambda \in \sigma_0(T)$). Conversely, if $\mathcal{R}(\lambda I - T)$ is closed and $\lambda \in \pi_{00}(T)$, then $\mathcal{R}(\lambda I - T)$ is closed and $\dim \mathcal{N}(\lambda I - T) < \infty$ (since $\lambda \in \sigma_{PF}(T)$), which means by Theorem 5.19 that $\lambda \in \sigma_0(T)$ (since $\lambda \in \sigma_{\mathrm{iso}}(T)$). Thus $\lambda \in \sigma_{\mathrm{iso}}(T) \cap \sigma_0(T) = \pi_0(T)$. $\qquad \square$

The set $\pi_0(T)$ of all Riesz points of T is sometimes referred to as the set of isolated eigenvalues of T of *finite algebraic multiplicity* (sometimes also called *normal eigenvalues* of T [55, p. 5], but we reserve this terminology for eigenvalues that satisfy the inclusion of Lemma 1.13(a)). The set $\pi_{00}(T)$ of all isolated eigenvalues of T of finite multiplicity is sometimes referred to as the set of isolated eigenvalues of T of *finite geometric multiplicity*.

The *Weyl spectrum* of an operator $T \in \mathcal{B}[\mathcal{H}]$ is the set

$$\sigma_w(T) = \bigcap\nolimits_{K \in \mathcal{B}_\infty[\mathcal{H}]} \sigma(T + K),$$

which is the largest part of $\sigma(T)$ that remains unchanged under compact perturbations. That is, $\sigma_w(T)$ *is the largest part of* $\sigma(T)$ *such that*

$$\sigma_w(T + K) = \sigma_w(T) \quad \text{for every} \quad K \in \mathcal{B}_\infty[\mathcal{H}].$$

Another characterization of $\sigma_w(T)$ will be given in Theorem 5.24.

Lemma 5.23. (a) *If* $T \in \mathcal{SF}$ *with* $\mathrm{ind}(T) \leq 0$, *then* $T \in \mathcal{F}_\ell$ *and there exists a compact operator* $K' \in \mathcal{B}_\infty[\mathcal{H}]$ *such that* $T + K'$ *is left invertible (i.e., such that there exists an operator* $A' \in \mathcal{B}[\mathcal{H}]$ *for which* $A'(T + K') = I$).

(b) *If* $T \in \mathcal{SF}$ *with* $\mathrm{ind}(T) \geq 0$, *then* $T \in \mathcal{F}_r$ *and there exists a compact operator* $K'' \in \mathcal{B}_\infty[\mathcal{H}]$ *such that* $T + K''$ *is right invertible (i.e., such that there exists an operator* $A'' \in \mathcal{B}[\mathcal{H}]$ *for which* $(T + K'')A'' = I$).

(c) *If* $T \in \mathcal{SF}$, *then* $\mathrm{ind}(T) = 0$ *if and only if there exists a compact operator* $K \in \mathcal{B}_\infty[\mathcal{H}]$ *such that* $T + K$ *is invertible (i.e., such that there exists an operator* $A \in \mathcal{B}[\mathcal{H}]$ *for which* $A(T + K) = (T + K)A = I$).

Proof. Take an operator T in $\mathcal{B}[\mathcal{H}]$.

Claim. If $T \in \mathcal{SF}$ with $\mathrm{ind}\,(T) \leq 0$, then $T \in \mathcal{F}_\ell$ and there is a compact (actually, a finite-rank) operator $K \in \mathcal{B}_\infty[\mathcal{H}]$ such that $\mathcal{N}(T + K) = \{0\}$.

Proof. Take $T \in \mathcal{SF}$. If $\mathrm{ind}\,(T) \leq 0$, then $\dim \mathcal{N}(T) - \dim \mathcal{N}(T^*) \leq 0$. Thus $\dim \mathcal{N}(T) \leq \dim \mathcal{N}(T^*)$ and $\dim \mathcal{N}(T) < \infty$ (so that $T \in \mathcal{F}_\ell$ by Theorem 5.1). Let $\{e_i\}_{i=1}^n$ be an orthonormal basis for $\mathcal{N}(T)$ and let B be an orthonormal basis for $\mathcal{N}(T^*) = \mathcal{R}(T)^\perp$ (cf. Lemma 1.4), whose cardinality is not less than n (since $\dim \mathcal{N}(T) \leq \dim \mathcal{N}(T^*)$). Take any orthonormal set $\{f_k\}_{k=1}^n \subseteq B$ and define a map $K \colon \mathcal{H} \to \mathcal{H}$ by

$$Kx = \sum\nolimits_{j=1}^n \langle x\,;e_j\rangle f_j \quad \text{for every} \quad x \in \mathcal{H},$$

which is clearly linear, and bounded as well (a contraction, actually). In fact, this is a finite-rank operator (thus compact); that is, K lies in $\mathcal{B}_0[\mathcal{H}] \subseteq \mathcal{B}_\infty[\mathcal{H}]$, whose range is included in $\mathcal{R}(T)^\perp$. Indeed,

$$\mathcal{R}(K) \subseteq \bigvee\{f_j\}_{j=1}^n \subseteq \bigvee B = \mathcal{N}(T^*) = \mathcal{R}(T)^\perp.$$

Take any $x \in \mathcal{N}(T)$ and consider its Fourier series expansion with respect to the orthonormal basis $\{e_i\}_{i=1}^n$, namely, $x = \sum_{=1}^n \langle x\,;e_i\rangle e_i$. Observe that

$$\|x\|^2 = \sum\nolimits_{j=1}^n |\langle x\,;e_i\rangle|^2 = \|Kx\|^2 \quad \text{for every} \quad x \in \mathcal{N}(T).$$

Now, if $x \in \mathcal{N}(T + K)$, then $Tx = -Kx$, and hence $Tx \in \mathcal{R}(T) \cap \mathcal{R}(K) \subseteq \mathcal{R}(T) \cap \mathcal{R}(T)^\perp = \{0\}$, so that $Tx = 0$ (i.e., $x \in \mathcal{N}(T)$). Thus $\|x\| = \|Kx\| = \|Tx\| = 0$, and hence $x = 0$. Therefore we get the claimed result:

$$\mathcal{N}(T + K) = \{0\}.$$

(a) If $T \in \mathcal{SF}$ with $\mathrm{ind}\,(T) \leq 0$, then $T \in \mathcal{F}_\ell$ and there exists $K \in \mathcal{B}_\infty[\mathcal{H}]$ such that $\mathcal{N}(T + K) = \{0\}$ by the preceding claim. Now $T + K \in \mathcal{F}_\ell$ according to Theorem 5.6, and so $\mathcal{R}(T + K)$ is closed by Theorem 5.1. Thus $T + K$ is left invertible by Lemma 5.8.

(b) If $T \in \mathcal{SF}$ with $\mathrm{ind}\,(T) \geq 0$, then $T^* \in \mathcal{SF}$ with $\mathrm{ind}\,(T^*) = -\mathrm{ind}\,(T) \leq 0$. Therefore, according to item (a), $T^* \in \mathcal{F}_\ell$, so that $T \in \mathcal{F}_r$, and there exists $K \in \mathcal{B}_\infty[\mathcal{H}]$ such that $T^* + K$ is left invertible. Hence $T + K^* = (T^* + K)^*$ is right invertible with $K^* \in \mathcal{B}_\infty[\mathcal{H}]$ (cf. Proposition 1.W).

(c) If $T \in \mathcal{SF}$ with $\mathrm{ind}\,(T) = 0$, then $T \in \mathcal{F}$ and there exists $K \in \mathcal{B}_\infty[\mathcal{H}]$ such that $\mathcal{N}(T + K) = \{0\}$ by the preceding claim. Since $\mathrm{ind}\,(T + K) = \mathrm{ind}\,(T) = 0$ (Theorem 5.6), we get $\mathcal{N}((T + K)^*) = \{0\}$. So $\mathcal{R}(T + K)^\perp = \{0\}$ (Lemma 1.4), and hence $\mathcal{R}(T + K) = \mathcal{H}$. Thus $T + K$ is invertible (Theorem 1.1). Conversely, if there exists $K \in \mathcal{B}_\infty[\mathcal{H}]$ such that $T + K$ is invertible, then $T + K$ is Weyl (recall that every invertible operator is Weyl), and so is T (Theorem 5.6); that is, $T \in \mathcal{F}$ with $\mathrm{ind}\,(T) = 0$. $\qquad\square$

According to the claim in the preceding proof, the statement of Lemma 5.23 holds if "compact" is replaced with "finite-rank" (see Remark 5.3(b)).

Theorem 5.24. (SCHECHTER THEOREM). *If $T \in \mathcal{B}[\mathcal{H}]$, then*

$$\sigma_w(T) = \sigma_e(T) \cup \bigcup\nolimits_{k \in \mathbb{Z} \setminus \{0\}} \sigma_k(T) = \sigma(T) \setminus \sigma_0(T).$$

Proof. Take an operator T in $\mathcal{B}[\mathcal{H}]$.

Claim. If $\lambda \in \sigma_0(T)$, then there is a $K \in \mathcal{B}_\infty[\mathcal{H}]$ such that $\lambda \notin \sigma(T + K)$.

Proof. If $\lambda \in \sigma_0(T)$, then $\lambda I - T \in \mathcal{SF}$ with $\mathrm{ind}(\lambda I - T) = 0$. Thus, according to Lemma 5.23(c), $\lambda I - (T + K)$ is invertible for some $K \in \mathcal{B}_\infty[\mathcal{H}]$, which implies that $\lambda \in \rho(T + K)$. This completes the proof of the claimed result.

Recall from Corollary 5.18 that

$$\sigma(T) = \sigma_e(T) \cup \bigcup\nolimits_{k \in \mathbb{Z} \setminus \{0\}} \sigma_k(T) \cup \sigma_0(T),$$

where the above sets are all pairwise disjoint, so that

$$\sigma_0(T) = \sigma(T) \setminus \left(\sigma_e(T) \cup \bigcup\nolimits_{k \in \mathbb{Z} \setminus \{0\}} \sigma_k(T) \right).$$

Take an arbitrary $\lambda \in \sigma(T)$. If λ lies in $\sigma_e(T) \cup \bigcup_{k \in \mathbb{Z} \setminus \{0\}} \sigma_k(T)$, then λ lies in $\sigma_e(T + K) \cup \bigcup_{k \in \mathbb{Z} \setminus \{0\}} \sigma_k(T + K)$ for every $K \in \mathcal{B}_\infty[\mathcal{H}]$, since

$$\sigma_e(T + K) = \sigma_e(T) \quad \text{and} \quad \sigma_k(T + K) = \sigma_k(T) \ \text{ for every } \ k \in \mathbb{Z} \setminus \{0\}$$

for every $K \in \mathcal{B}_\infty[\mathcal{H}]$ (cf. Remarks 5.15(c) and 5.17(c)). On the other hand, if $\lambda \in \sigma_0(T)$, then the preceding claim ensures that there exists $K \in \mathcal{B}_\infty[\mathcal{H}]$ for which $\lambda \notin \sigma(T + K)$. Therefore, as the largest part of $\sigma(T)$ that remains invariant under compact perturbation is $\sigma_w(T)$,

$$\sigma_w(T) = \sigma_e(T) \cup \bigcup\nolimits_{k \in \mathbb{Z} \setminus \{0\}} \sigma_k(T) = \sigma(T) \setminus \sigma_0(T). \qquad \square$$

Since $\sigma_w(T) \subseteq \sigma(T)$ by the very definition of $\sigma_w(T)$, we get

$$\sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma(T).$$

Chronologically, Theorem 5.24 precedes the spectral picture of Corollary 5.18 [81]. Since $\sigma_k(T) \subseteq \sigma(T) \setminus \sigma_e(T)$ for all $k \in \mathbb{Z}$, it also follows that

$$\sigma_w(T) \setminus \sigma_e(T) = \bigcup\nolimits_{k \in \mathbb{Z} \setminus \{0\}} \sigma_k(T),$$

which is the collection of all holes of the essential spectrum. Thus

$$\sigma_e(T) = \sigma_w(T) \quad \Longleftrightarrow \quad \bigcup\nolimits_{k \in \mathbb{Z} \setminus \{0\}} \sigma_k(T) = \varnothing$$

(i.e., if and only if the essential spectrum has no holes). Since $\sigma_w(T)$ and $\sigma_0(T)$ are both subsets of $\sigma(T)$, it follows by Theorem 5.24 that $\sigma_0(T)$ is the complement of $\sigma_w(T)$ in $\sigma(T)$,

$$\sigma_0(T) = \sigma(T)\backslash\sigma_w(T),$$

and therefore $\{\sigma_w(T), \sigma_0(T)\}$ forms a partition of the spectrum $\sigma(T)$:

$$\sigma(T) = \sigma_w(T) \cup \sigma_0(T) \quad \text{and} \quad \sigma_w(T) \cap \sigma_0(T) = \varnothing.$$

Thus

$$\sigma_w(T) = \sigma(T) \quad \Longleftrightarrow \quad \sigma_0(T) = \varnothing,$$

and so, by Theorem 5.24 again,

$$\sigma_e(T) = \sigma_w(T) = \sigma(T) \quad \Longleftrightarrow \quad \bigcup_{k\in\mathbb{Z}}\sigma_k(T) = \varnothing.$$

Moreover, the Weyl spectrum $\sigma_w(T)$ is always a compact set in $\mathbb{C}$ because it is, by definition, the intersection $\bigcap_{K\in\mathcal{B}_\infty[\mathcal{H}]} \sigma(T + K)$ of compact sets in $\mathbb{C}$, and so $\sigma_0(T)$ is closed in $\mathbb{C}$ if and only if $\sigma_0(T) = \pi_0(T) \subseteq \sigma_{\mathrm{iso}}(T)$ (why?). Here is a third characterization of the Weyl spectrum of $T \in \mathcal{B}[\mathcal{H}]$.

Corollary 5.25. *For every operator $T \in \mathcal{B}[\mathcal{H}]$,*

$$\sigma_w(T) = \big\{\lambda \in \mathbb{C}: \lambda I - T \in \mathcal{B}[\mathcal{H}]\backslash\mathcal{W}\big\}.$$

Proof. If $\lambda \in \rho(T)$, then $\lambda T - T$ is invertible, and so $\lambda T - T \in \mathcal{W}$ (invertible operators are Weyl). Therefore, if $\lambda T - T \notin \mathcal{W}$, then $\lambda \in \sigma(T)$, and the result follows from Theorem 5.24 and the definition of $\sigma_0(T)$:

$$\sigma_w(T) = \sigma(T)\backslash\sigma_0(T) \quad \text{and} \quad \sigma_0(T) = \{\lambda \in \sigma(T): \lambda I - T \in \mathcal{W}\}. \qquad \square$$

Then the Weyl spectrum $\sigma_w(T)$ is the set of all scalars λ for which $\lambda I - T$ is not a Weyl operator (i.e., for which $\lambda I - T$ is not a Fredholm operator of index zero). Since an operator lies in $\mathcal{W}$ together with its adjoint, it follows by Corollary 5.25 that $\lambda \in \sigma_w(T)$ if and only if $\overline{\lambda} \in \sigma_w(T^*)$:

$$\sigma_w(T) = \sigma_w(T^*)^*.$$

Theorem 5.24 also gives us another characterization of the set $\mathcal{W}$ of all Weyl operators from $\mathcal{B}[\mathcal{H}]$ in terms of the set $\sigma_0(T) = \sigma(T)\backslash\sigma_w(T)$.

Corollary 5.26.

$$\mathcal{W} = \big\{T \in \mathcal{B}[\mathcal{H}]: 0 \in \rho(T) \cup \sigma_0(T)\big\} = \big\{T \in \mathcal{F}: 0 \in \rho(T) \cup \sigma_0(T)\big\}.$$

Proof. Take $T \in \mathcal{B}[\mathcal{H}]$. If $T \in \mathcal{W}$, then $T \in \mathcal{F}$ and $T + K$ is invertible for some $K \in \mathcal{B}_\infty[\mathcal{H}]$ by Lemma 5.23(c), which means that $0 \in \rho(T + K)$ or, equivalently, $0 \notin \sigma(T + K)$, and so $0 \notin \sigma_w(T + K)$. Thus $0 \notin \sigma_w(T)$ (cf. definition

of the Weyl spectrum that precedes Lemma 5.23), and hence $0 \in \rho(T) \cup \sigma_0(T)$ by Theorem 5.24. The converse is (almost) trivial: if $0 \in \rho(T) \cup \sigma_0(T)$, then either $0 \in \rho(T)$, so that $T \in \mathcal{G}[\mathcal{H}] \subset \mathcal{W}$ by Remark 5.7(b); or $0 \in \sigma_0(T)$, which means that $0 \in \sigma(T)$ and $T \in \mathcal{W}$ by the very definition of $\sigma_0(T)$. This proves the first identity, which implies the second one since $\mathcal{W} \subseteq \mathcal{F}$. $\square$

Remark 5.27. (a) Finite Dimensional. Take any operator $T \in \mathcal{B}[\mathcal{H}]$. Since $\sigma_e(T) \subseteq \sigma_w(T)$, it follows by Remark 5.15(a) that

$$\sigma_w(T) = \varnothing \quad \Longrightarrow \quad \dim \mathcal{H} < \infty$$

and, by Remark 5.3(c) and Corollary 5.25,

$$\dim \mathcal{H} < \infty \quad \Longrightarrow \quad \sigma_w(T) = \varnothing.$$

Thus

$$\sigma_w(T) \neq \varnothing \quad \Longleftrightarrow \quad \dim \mathcal{H} = \infty.$$

(b) More on Finite Dimensional. Let $T \in \mathcal{B}[\mathcal{H}]$ be an operator on a finite-dimensional space. Then $\mathcal{B}[\mathcal{H}] = \mathcal{W}$ by Remark 5.3(c) so that $\sigma_0(T) = \sigma(T)$ by the definition of $\sigma_0(T)$. Moreover, Corollary 2.9 ensures that $\sigma(T) = \sigma_{\mathrm{iso}}(T)$ (since $\#\sigma(T) < \infty$), and $\sigma(T) = \sigma_{PF}(T)$ (since $\dim \mathcal{H} < \infty$). Hence,

$$\dim \mathcal{H} < \infty \quad \Longrightarrow \quad \sigma(T) = \sigma_{PF}(T) = \sigma_{\mathrm{iso}}(T) = \sigma_0(T) = \pi_0(T) = \pi_{00}(T)$$

and so, either by Theorem 5.24 or by item (a) and Remark 5.15(a),

$$\dim \mathcal{H} < \infty \quad \Longrightarrow \quad \sigma_e(T) = \sigma_w(T) = \varnothing.$$

(c) Fredholm Alternative. The Fredholm Alternative has been restated again and again in Corollary 1.20, Theorem 2.18, Remark 5.3(d), Remark 5.7(b), and Remark 5.17(a). Here is an ultimate restatement of it. If K is compact (i.e., $K \in \mathcal{B}_\infty[\mathcal{H}]$), then $\sigma(K)\backslash\{0\} = \sigma_0(K)\backslash\{0\}$ (Remark 5.17(a)). But $\sigma(K)\backslash\{0\} \subseteq \sigma_{\mathrm{iso}}(K)$ by Corollary 2.20, and $\pi_0(K) = \sigma_{\mathrm{iso}}(K) \cap \sigma_0(K)$. Therefore (compare with Theorem 2.18 and Remark 5.17(a)),

$$K \in \mathcal{B}_\infty[\mathcal{H}] \quad \Longrightarrow \quad \sigma(K)\backslash\{0\} = \pi_0(K)\backslash\{0\}.$$

Since $\pi_0(K) = \sigma_{\mathrm{iso}}(K) \cap \sigma_0(K) \subseteq \sigma_{\mathrm{iso}}(K) \cap \sigma_{PF}(K) = \pi_{00}(K) \subseteq \sigma_{PF}(T) \subseteq \sigma_P(T)$,

$$\sigma(K)\backslash\{0\} = \sigma_{PF}(K)\backslash\{0\} = \pi_{00}(K)\backslash\{0\} = \pi_0(K)\backslash\{0\}.$$

(d) More on Compact Operators. Take a compact operator $K \in \mathcal{B}_\infty[\mathcal{H}]$. Remark 5.3(d) (Fredholm Alternative) says that, if $\lambda \neq 0$, then $\lambda I - K \in \mathcal{W}$. Thus, according to Corollary 5.25, $\sigma_w(K)\backslash\{0\} = \varnothing$. If $\dim \mathcal{H} = \infty$, then $\sigma_w(K) = \{0\}$ by item (a). Since $\sigma_e(K) \subseteq \sigma_w(K)$, it follows by Remark 5.15(a) that, if $\dim \mathcal{H} = \infty$, then $\varnothing \neq \sigma_e(K) \subseteq \{0\}$. Hence,

$$K \in \mathcal{B}_\infty[\mathcal{H}] \text{ and } \dim \mathcal{H} = \infty \quad \Longrightarrow \quad \sigma_e(K) = \sigma_w(K) = \{0\}.$$

Therefore, by the Fredholm Alternative of item (c),

$$K \in \mathcal{B}_\infty[\mathcal{H}] \ \text{ and } \ \dim \mathcal{H} = \infty \quad \Longrightarrow \quad \sigma(K) = \pi_0(K) \cup \{0\}, \quad \sigma_w(K) = \{0\}.$$

(e) NORMAL OPERATORS. Recall that T is normal if and only if $\lambda I - T$ is normal. Since every normal Fredholm operator is Weyl (see Remark 5.7(b)), and according to Corollaries 5.12 and 5.25, it follows that

$$T \text{ normal} \quad \Longrightarrow \quad \sigma_e(T) = \sigma_w(T).$$

Take any $T \in \mathcal{B}[\mathcal{H}]$ and consider the following expressions (cf. Theorem 5.24 or Corollary 5.25, and the very definitions of $\sigma_0(T)$, $\pi_0(T)$, and $\pi_{00}(T)$).

$$\sigma_0(T) = \sigma(T) \backslash \sigma_w(T), \quad \pi_0(T) = \sigma_{\mathrm{iso}}(T) \cap \sigma_0(T), \quad \pi_{00}(T) = \sigma_{\mathrm{iso}}(T) \cap \sigma_{PF}(T).$$

The equivalent assertions in the next corollary define an important class of operators. An operator for which any of those equivalent assertions holds is said to satisfy *Weyl's Theorem*. This will be discussed later in Section 5.5.

Corollary 5.28. *The assertions below are pairwise equivalent.*

(a) $\sigma_0(T) = \pi_{00}(T)$.

(b) $\sigma(T) \backslash \pi_{00}(T) = \sigma_w(T)$.

Moreover, if
$$\sigma_0(T) = \sigma_{\mathrm{iso}}(T)$$

(i.e., if $\sigma_w(T) = \sigma_{\mathrm{acc}}(T)$), then the above equivalent assertions hold true. Conversely, if any of the above equivalent assertions holds true, then

$$\pi_0(T) = \pi_{00}(T).$$

Proof. By Theorem 5.24, $\sigma(T) \backslash \sigma_0(T) = \sigma_w(T)$, and so (a) implies (b). If (b) holds, then $\sigma_0(T) = \sigma(T) \backslash \sigma_w(T) = \sigma(T) \backslash \big(\sigma(T) \backslash \pi_{00}(T) \big) = \pi_{00}(T)$, because $\pi_{00}(T) \subseteq \sigma(T)$, and so (b) implies (a). Moreover, if $\sigma_0(T) = \sigma_{\mathrm{iso}}(T)$ (or, if their complements in $\sigma(T)$ coincide), then (a) holds because $\sigma_0(T) \subseteq \sigma_{PF}(T)$. Conversely, if (a) holds, then $\pi_0(T) = \sigma_{\mathrm{iso}}(T) \cap \sigma_0(T) = \sigma_{\mathrm{iso}}(T) \cap \pi_{00}(T) = \pi_{00}(T)$ because $\pi_{00}(T) \subseteq \sigma_{\mathrm{iso}}(T)$. $\qquad \square$

5.4 Ascent, Descent, and Browder Spectrum

Take any $T \in \mathcal{B}[\mathcal{H}]$ (or, more generally, take any linear transformation of linear space into itself). Let $\mathbb{N}_0$ be the set of all nonnegative integers, and consider the nonnegative integral powers T^n of T. It is readily verified that, for $n \in \mathbb{N}_0$,

$$\mathcal{N}(T^n) \subseteq \mathcal{N}(T^{n+1}) \quad \text{and} \quad \mathcal{R}(T^{n+1}) \subseteq \mathcal{R}(T^n),$$

which means that $\{\mathcal{N}(T^n)\}$ and $\{\mathcal{R}(T^n)\}$ are nondecreasing and nonincreasing (in the inclusion ordering) sequences of subsets of $\mathcal{H}$, respectively.

Lemma 5.29. *Let n_0 be an arbitrary integer in $\mathbb{N}_0$.*

(a) *If $\mathcal{N}(T^{n_0+1}) = \mathcal{N}(T^{n_0})$, then $\mathcal{N}(T^{n+1}) = \mathcal{N}(T^n)$ for every $n \geq n_0$.*

(b) *If $\mathcal{R}(T^{n_0+1}) = \mathcal{R}(T^{n_0})$, then $\mathcal{R}(T^{n+1}) = \mathcal{R}(T^n)$ for every $n \geq n_0$.*

Proof. (a) Rewrite the statements in (a) as follows.

$$\mathcal{N}(T^{n_0+1}) = \mathcal{N}(T^{n_0}) \quad \Longrightarrow \quad \mathcal{N}(T^{n_0+k+1}) = \mathcal{N}(T^{n_0+k}) \ \text{ for every } \ k \geq 0.$$

The claimed result holds trivially for $k = 0$. Suppose it holds for some $k \geq 0$. Take an arbitrary $x \in \mathcal{N}(T^{n_0+k+2})$ so that $T^{n_0+k+1}(Tx) = T^{n_0+k+2}x = 0$. Thus $Tx \in \mathcal{N}(T^{n_0+k+1}) = \mathcal{N}(T^{n_0+k})$, and so $T^{n_0+k+1}x = T^{n_0+k}(Tx) = 0$, which implies that $x \in \mathcal{N}(T^{n_0+k+1})$. Hence, $\mathcal{N}(T^{n_0+k+2}) \subseteq \mathcal{N}(T^{n_0+k+1})$. But $\mathcal{N}(T^{n_0+k+1}) \subseteq \mathcal{N}(T^{n_0+k+2})$ since $\{\mathcal{N}(T^n)\}$ is nondecreasing. Therefore $\mathcal{N}(T^{n_0+k+2}) = \mathcal{N}(T^{n_0+k+1})$, and the claimed result holds for $k+1$ whenever it holds for k, which completes the proof of (a) by induction.

(b) Rewrite the statements in (b) as follows.

$$\mathcal{R}(T^{n_0+1}) = \mathcal{R}(T^{n_0}) \quad \Longrightarrow \quad \mathcal{R}(T^{n_0+k+1}) = \mathcal{R}(T^{n_0+k}) \ \text{ for every } \ k \geq 1.$$

The claimed result holds trivially for $k = 0$. Suppose it holds for some integer $k \geq 0$. Take an arbitrary $y \in \mathcal{R}(T^{n_0+k+1})$ so that $y = T^{n_0+k+1}x = T(T^{n_0+k}x)$ for some $x \in \mathcal{H}$, and hence $y = Tu$ for some $u \in \mathcal{R}(T^{n_0+k})$. If $\mathcal{R}(T^{n_0+k}) = \mathcal{R}(T^{n_0+k+1})$, then $u \in \mathcal{R}(T^{n_0+k+1})$, and so $y = T(T^{n_0+k+1}v)$ for some $v \in \mathcal{H}$. Thus $y \in \mathcal{R}(T^{n_0+k+2})$. Therefore, $\mathcal{R}(T^{n_0+k+1}) \subseteq \mathcal{R}(T^{n_0+k+2})$. Since the sequence $\{\mathcal{R}(T^n)\}$ is nonincreasing, it follows that $\mathcal{R}(T^{n_0+k+2}) \subseteq \mathcal{R}(T^{n_0+k+1})$. Hence $\mathcal{R}(T^{n_0+k+2}) = \mathcal{R}(T^{n_0+k+1})$. Thus the claimed result holds for $k+1$ whenever it holds for k, which completes the proof of (b) by induction. $\qquad\square$

Let $\overline{\mathbb{N}}_0 = \mathbb{N}_0 \cup \{+\infty\}$ denote the set of all extended nonnegative integers with its natural (extended) ordering. The *ascent* and *descent* of an operator $T \in \mathcal{B}[\mathcal{H}]$ are defined as follows. The *ascent* of T is the least (extended) nonnegative integer such that $\mathcal{N}(T^{n+1}) = \mathcal{N}(T^n)$, that is,

$$\mathrm{asc}\,(T) = \min\big\{n \in \overline{\mathbb{N}}_0 \colon \mathcal{N}(T^{n+1}) = \mathcal{N}(T^n)\big\},$$

and the *descent* of T is the least (extended) nonnegative integer such that $\mathcal{R}(T^{n+1}) = \mathcal{R}(T^n)$, that is

$$\mathrm{dsc}\,(T) = \min\big\{n \in \overline{\mathbb{N}}_0 \colon \mathcal{R}(T^{n+1}) = \mathcal{R}(T^n)\big\}.$$

Note that the ascent of T is null if and only if T is injective, and the descent of T is null if and only if T is surjective. Indeed, since $\mathcal{N}(I) = \{0\}$ and $\mathcal{R}(I) = \mathcal{H}$,

$$\mathrm{asc}\,(T) = 0 \quad \Longleftrightarrow \quad \mathcal{N}(T) = \{0\},$$

$$\mathrm{dsc}\,(T) = 0 \quad \Longleftrightarrow \quad \mathcal{R}(T) = \mathcal{H}.$$

Again, the notion of ascent and descent holds for every linear transformation of linear space into itself, and so does the next result.

Lemma 5.30. *Take any operator $T \in \mathcal{B}[\mathcal{H}]$.*

(a) *If $\mathrm{asc}\,(T) < \infty$ and $\mathrm{dsc}\,(T) = 0$, then $\mathrm{asc}\,(T) = 0$.*

(b) *If $\mathrm{asc}\,(T) < \infty$ and $\mathrm{dsc}\,(T) < \infty$, then $\mathrm{asc}\,(T) = \mathrm{dsc}\,(T)$.*

Proof. (a) Suppose $\mathrm{dsc}\,(T) = 0$ (i.e., suppose $\mathcal{R}(T) = \mathcal{H}$). If $\mathrm{asc}\,(T) \neq 0$ (i.e., if $\mathcal{N}(T) \neq \{0\}$), then take $0 \neq x_1 \in \mathcal{N}(T) \cap \mathcal{R}(T)$ and x_2, x_3 in $\mathcal{R}(T) = \mathcal{H}$ such that $x_1 = Tx_2$ and $x_2 = Tx_3$, and so $x_1 = T^2 x_3$. Continuing this way, we can construct a sequence $\{x_n\}_{n \geq 1}$ of vectors in $\mathcal{H} = \mathcal{R}(T)$ such that $x_n = Tx_{n+1}$ and $0 \neq x_1 = T^n x_{n+1}$ lies in $\mathcal{N}(T)$, and so $T^{n+1} x_{n+1} = 0$. This implies that $x_{n+1} \in \mathcal{N}(T^{n+1}) \backslash \mathcal{N}(T^n)$ for each $n \geq 1$, and hence $\mathrm{asc}\,(T) = \infty$ by Lemma 5.29. Summing up: If $\mathrm{dsc}\,(T) = 0$, then $\mathrm{asc}\,(T) \neq 0$ implies $\mathrm{asc}\,(T) = \infty$.

(b) Set $m = \mathrm{dsc}\,(T)$, so that $\mathcal{R}(T^m) = \mathcal{R}(T^{m+1})$, and set $S = T|_{\mathcal{R}(T^m)}$. It is readily verified by using Proposition 1.E that $\mathcal{R}(T^m)$ is T-invariant, and so $S \in \mathcal{B}[\mathcal{R}(T^m)]$. Note that $\mathcal{R}(S) = S(\mathcal{R}(T^m)) = \mathcal{R}(ST^m) = \mathcal{R}(T^{m+1}) = \mathcal{R}(T^m)$. Thus S is surjective, which means that $\mathrm{dsc}\,(S) = 0$. Moreover, since $\mathrm{asc}\,(S) < \infty$ (because $\mathrm{asc}\,(T) < \infty$), it follows by (a) that $\mathrm{asc}\,(S) = 0$. That is, $\mathcal{N}(S) = \{0\}$. Take $x \in \mathcal{N}(T^{m+1})$ and set $y = T^m x$ in $\mathcal{R}(T^m)$, so that $Sy = T^{m+1}x = 0$. Thus $y = 0$, and hence $x \in \mathcal{N}(T^m)$. Then $\mathcal{N}(T^{m+1}) \subseteq \mathcal{N}(T^m)$, and so $\mathcal{N}(T^{m+1}) = \mathcal{N}(T^m)$ since $\{\mathcal{N}(T^n)\}$ is nondecreasing. This implies that $\mathrm{asc}\,(T) \leq m$ by Lemma 5.29. To prove the reverse inequality, suppose $m \neq 0$ (otherwise apply (a)) and take z in $\mathcal{R}(T^{m-1}) \backslash \mathcal{R}(T^m)$ so that $z = T^{m-1}u$ and $Tz = T(T^{m-1}u) = T^m u$ lies in $\mathcal{R}(T^m)$, for some u in $\mathcal{H}$. Since $T^m(\mathcal{R}(T^m)) = \mathcal{R}(T^{2m}) = \mathcal{R}(T^m)$, we get $Tz = T^m v$ for some v in $\mathcal{R}(T^m)$. Now note that $T^m(u - v) = Tz - Tz = 0$ and $T^{m-1}(u-v) = z - T^{m-1}v \neq 0$ (reason: $T^{m-1}v \in \mathcal{R}(T^{2m-1}) = \mathcal{R}(T^m)$ since $v \in \mathcal{R}(T^m)$, and $z \notin \mathcal{R}(T^m)$). Therefore, $(u - v) \in \mathcal{N}(T^m) \backslash \mathcal{N}(T^{m-1})$, and so $\mathrm{asc}\,(T) \geq m$. Outcome: If $\mathrm{dsc}\,(T) = m$ and $\mathrm{asc}\,(T) < \infty$, then $\mathrm{asc}\,(T) = m$. $\square$

Lemma 5.31. *If $T \in \mathcal{F}$, then $\mathrm{asc}\,(T) = \mathrm{dsc}\,(T^*)$ and $\mathrm{dsc}\,(T) = \mathrm{asc}\,(T^*)$.*

Proof. Take any $T \in \mathcal{B}[\mathcal{H}]$. We shall use freely the relations between range and kernel, involving adjoints and orthogonal complements, of Lemma 1.4.

Claim 1. $\mathrm{asc}\,(T) < \infty$ if and only if $\mathrm{dsc}\,(T^*) < \infty$.
$\quad\quad\quad$ $\mathrm{dsc}\,(T) < \infty$ if and only if $\mathrm{asc}\,(T^*) < \infty$.

Proof. Take an arbitrary $n \in \mathbb{N}_0$. If $\mathrm{asc}\,(T) = \infty$, then $\mathcal{N}(T^n) \subset \mathcal{N}(T^{n+1})$ so that $\mathcal{N}(T^{n+1})^{\perp} \subset \mathcal{N}(T^n)^{\perp}$ or, equivalently, $\mathcal{R}(T^{*(n+1)})^- \subset \mathcal{R}(T^{*n})^-$, which implies that $\mathcal{R}(T^{*(n+1)}) \subset \mathcal{R}(T^{*n})$, and therefore $\mathrm{dsc}\,(T^*) = \infty$. Dually, if $\mathrm{asc}\,(T^*) = \infty$, then $\mathrm{dsc}\,(T) = \infty$. That is,

$$\mathrm{asc}\,(T) = \infty \implies \mathrm{dsc}\,(T^*) = \infty \quad \text{and} \quad \mathrm{asc}\,(T^*) = \infty \implies \mathrm{dsc}\,(T) = \infty.$$

If $\mathrm{dsc}\,(T) = \infty$, then $\mathcal{R}(T^{n+1}) \subset \mathcal{R}(T^n)$ so that $\mathcal{R}(T^n)^\perp \subset \mathcal{R}(T^{n+1})^\perp$ or, equivalently, $\mathcal{N}(T^{*n}) \subset \mathcal{N}(T^{*(n+1)})$, and therefore $\mathrm{asc}\,(T^*) = \infty$. Dually, if $\mathrm{dsc}\,(T^*) = \infty$, then $\mathrm{asc}\,(T) = \infty$. That is,

$$\mathrm{dsc}\,(T) = \infty \implies \mathrm{asc}\,(T^*) = \infty \quad \text{and} \quad \mathrm{dsc}\,(T^*) = \infty \implies \mathrm{asc}\,(T) = \infty.$$

Summing up: $\mathrm{asc}\,(T) = \infty$ if and only if $\mathrm{dsc}\,(T^*) = \infty$ and $\mathrm{dsc}\,(T) = \infty$ if and only if $\mathrm{asc}\,(T^*) = \infty$, which completes the proof of Claim 1.

Claim 2. If $\mathrm{dsc}\,(T) < \infty$, then $\mathrm{asc}\,(T^*) \leq \mathrm{dsc}\,(T)$.
If $\mathrm{dsc}\,(T^*) < \infty$, then $\mathrm{asc}\,(T) \leq \mathrm{dsc}\,(T^*)$.

Proof. If $\mathrm{dsc}\,(T) < \infty$, then set $n_0 = \mathrm{dsc}\,(T) \in \mathbb{N}_0$ and take any integer $n \geq n_0$. Thus $\mathcal{R}(T^n) = \mathcal{R}(T^{n_0})$, and hence $\mathcal{R}(T^n)^- = \mathcal{R}(T^{n_0})^-$ so that $\mathcal{N}(T^{*n})^\perp = \mathcal{N}(T^{*n_0})^\perp$, which implies $\mathcal{N}(T^{*n}) = \mathcal{N}(T^{*n_0})$. Therefore, $\mathrm{asc}\,(T^*) \leq n_0$, and so $\mathrm{asc}\,(T^*) \leq \mathrm{dsc}\,(T)$. Dually, if $\mathrm{dsc}\,(T^*) < \infty$, then $\mathrm{asc}\,(T) \leq \mathrm{dsc}\,(T^*)$. This completes the proof of Claim 2.

Claim 3. If $\mathrm{asc}\,(T) < \infty$ and $\mathcal{R}(T^n)$ is closed for every $n \geq \mathrm{asc}\,(T)$, then $\mathrm{dsc}\,(T^*) \leq \mathrm{asc}\,(T)$.
If $\mathrm{asc}\,(T^*) < \infty$ and $\mathcal{R}(T^n)$ is closed for every $n \geq \mathrm{asc}\,(T^*)$, then $\mathrm{dsc}\,(T) \leq \mathrm{asc}\,(T^*)$.

Proof. If $\mathrm{asc}\,(T) < \infty$, then set $n_0 = \mathrm{asc}\,(T) \in \mathbb{N}_0$ and take an arbitrary integer $n \geq n_0$. Thus $\mathcal{N}(T^n) = \mathcal{N}(T^{n_0})$ or, equivalently, $\mathcal{R}(T^{*n})^\perp = \mathcal{R}(T^{*n_0})^\perp$, so that $\mathcal{R}(T^{*n_0})^- = \mathcal{R}(T^{*n})^-$. Suppose $\mathcal{R}(T^n)$ is closed. Thus $\mathcal{R}(T^{*n}) = \mathcal{R}(T^{n*})$ is closed (Lemma 1.5). Then $\mathcal{R}(T^{*n}) = \mathcal{R}(T^{*n_0})$, and therefore $\mathrm{dsc}\,(T^*) \leq n_0$, so that $\mathrm{dsc}\,(T^*) \leq \mathrm{asc}\,(T)$. Dually, if $\mathrm{asc}\,(T^*) < \infty$ and $\mathcal{R}(T^n)$ is closed (i.e., $\mathcal{R}(T^{*n})$ is closed — Lemma 1.5) for every integer $n \geq \mathrm{asc}\,(T^*)$, then it follows that $\mathrm{dsc}\,(T) \leq \mathrm{asc}\,(T^*)$; completing the proof of Claim 3.

Outcome: If $\mathcal{R}(T^n)$ is closed for every $n \in \mathbb{N}_0$ (so that we can apply Claim 3), then it follows by Claims 1, 2, and 3 that

$$\mathrm{asc}\,(T) = \mathrm{dsc}\,(T^*) \quad \text{and} \quad \mathrm{dsc}\,(T) = \mathrm{asc}\,(T^*).$$

In particular, if $T \in \mathcal{F}$, then $T^n \in \mathcal{F}$ by Corollary 5.5, which implies that $\mathcal{R}(T^n)$ is closed for every $n \in \mathbb{N}_0$ by Corollary 5.2. Therefore, if T is Fredholm, then $\mathrm{asc}\,(T) = \mathrm{dsc}\,(T^*)$ and $\mathrm{dsc}\,(T) = \mathrm{asc}\,(T^*)$. $\qquad\square$

A *Browder operator* is a Fredholm operator with finite ascent and finite descent. Let $\mathcal{B}$ denote the class of all Browder operators from $\mathcal{B}[\mathcal{H}]$:

$$\mathcal{B} = \big\{T \in \mathcal{F}: \mathrm{asc}\,(T) < \infty \text{ and } \mathrm{dsc}\,(T) < \infty\big\};$$

equivalently, according to Lemma 5.30(b),

$$\mathcal{B} = \big\{T \in \mathcal{F}: \mathrm{asc}\,(T) = \mathrm{dsc}\,(T) < \infty\big\}$$

(i.e., $\mathcal{B} = \{T \in \mathcal{F} \colon \operatorname{asc}(T) = \operatorname{dsc}(T) = m$ for some $m \in \mathbb{N}_0\}$). Thus

$$\mathcal{F} \backslash \mathcal{B} = \{T \in \mathcal{F} \colon \operatorname{asc}(T) = \infty \text{ or } \operatorname{dsc}(T) = \infty\}$$

and, by Lemma 5.31,

$$T \in \mathcal{B} \quad \text{if and only if} \quad T^* \in \mathcal{B}.$$

Two linear manifolds of a linear space $\mathcal{X}$, say $\mathcal{R}$ and $\mathcal{N}$, are said to be *algebraic complements* of each other (or *complementary linear manifolds*) if they sum the whole space and are algebraically disjoint; that is, if

$$\mathcal{R} + \mathcal{N} = \mathcal{X} \quad \text{and} \quad \mathcal{R} \cap \mathcal{N} = \{0\}.$$

A pair of subspaces (i.e., closed linear manifolds) of a normed space that are algebraic complements of each other are called *complementary subspaces*.

Lemma 5.32. *If $T \in \mathcal{B}[\mathcal{H}]$ with $\operatorname{asc}(T) = \operatorname{dsc}(T) = m$ for some $m \in \mathbb{N}_0$, then $\mathcal{R}(T^m)$ and $\mathcal{N}(T^m)$ are algebraic complements of each other.*

Proof. Take any $T \in \mathcal{B}[\mathcal{H}]$. (In fact, as will become clear during the proof, this is a purely algebraic result which holds for every linear transformation of a linear space into itself with coincident finite ascent and descent.)

Claim 1. If $\operatorname{asc}(T) = m$, then $\mathcal{R}(T^m) \cap \mathcal{N}(T^m) = \{0\}$.

Proof. If $y \in \mathcal{R}(T^m) \cap \mathcal{N}(T^m)$, then $y = T^m x$ for some $x \in \mathcal{H}$ and $T^m y = 0$, so that $T^{2m} x = T^m(T^m x) = T^m y = 0$; that is, $x \in \mathcal{N}(T^{2m}) = \mathcal{N}(T^m)$ since $\operatorname{asc}(T) = m$. Hence $y = T^m x = 0$, proving Claim 1.

Claim 2. If $\operatorname{dsc}(T) = m$, then $\mathcal{R}(T^m) + \mathcal{N}(T^m) = \mathcal{H}$.

Proof. If $\operatorname{dsc}(T) = m$, then $\mathcal{R}(T^m) = \mathcal{R}(T^{m+1})$. Set $S = T|_{\mathcal{R}(T^m)}$. Again, by using Proposition 1.E, it is readily verified that $\mathcal{R}(T^m)$ is T-invariant, and so $S \in \mathcal{B}[\mathcal{R}(T^m)]$ with $\mathcal{R}(S) = S(\mathcal{R}(T^m)) = \mathcal{R}(ST^m) = \mathcal{R}(T^{m+1}) = \mathcal{R}(T^m)$. Then S is surjective, and hence S^m is surjective too (since $\{\mathcal{R}(S^n)\}$ is non-decreasing). Therefore, $\mathcal{R}(S^m) = \mathcal{R}(T^m)$. Take an arbitrary vector $x \in \mathcal{H}$. Thus $T^m x \in \mathcal{R}(T^m) = \mathcal{R}(S^m)$ so that there exists $u \in \mathcal{R}(T^m)$ such that $S^m u = T^m x$. Since $S^m u = T^m u$ (because $S^m = (T|_{\mathcal{R}(T^m)})^m = T^m|_{\mathcal{R}(T^m)}$), it follows that $T^m u = T^m x$, and so $v = x - u$ is in $\mathcal{N}(T^m)$. Then $x = u + v$ lies in $\mathcal{R}(T^m) + \mathcal{N}(T^m)$. Hence $\mathcal{H} \subseteq \mathcal{R}(T^m) + \mathcal{N}(T^m)$, proving Claim 2.   □

Theorem 5.33. *Take $T \in \mathcal{B}[\mathcal{H}]$ and consider the following assertions.*

(a) *T is Browder but not invertible (i.e., $T \in \mathcal{B}$ and $T \notin \mathcal{G}[\mathcal{H}]$).*

(b) *$T \in \mathcal{F}$ is such that $\mathcal{R}(T^m)$ and $\mathcal{N}(T^m)$ are complementary subspaces for some $m \in \mathbb{N}$.*

(c) *$T \in \mathcal{W}$.*

Claim: (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c).

Proof. (a)$\Rightarrow$(b). Recall that $T \in \mathcal{B}[\mathcal{H}]$ is invertible (equivalently, $\mathcal{R}(T) = \mathcal{H}$ and $\mathcal{N}(T) = \{0\}$) if and only if $T \in \mathcal{G}[\mathcal{H}]$ (i.e., $T \in \mathcal{B}[\mathcal{H}]$ has a bounded inverse) by Theorem 1.1. Moreover, every invertible operator is Fredholm (by the definition of Fredholm operators), and $\mathcal{R}(T) = \mathcal{H}$ and $\mathcal{N}(T) = \{0\}$ if and only if $\mathrm{asc}\,(T) = \mathrm{dsc}\,(T) = 0$ (by the definition of ascent and descent). Thus every invertible operator is Browder, which in turn is Fredholm,

$$\mathcal{G}[\mathcal{H}] \subset \mathcal{B} \subset \mathcal{F},$$

and it is readily verified that the inclusions are, in fact, proper. Therefore, if $T \in \mathcal{B}$ and $T \notin \mathcal{G}[\mathcal{H}]$, then $\mathrm{asc}\,(T) = \mathrm{dsc}\,(T) = m$ for some integer $m \geq 1$, and so $\mathcal{N}(T^m)$ and $\mathcal{R}(T^m)$ are complementary linear manifolds of $\mathcal{H}$ by Lemma 5.32. If $T \in \mathcal{B}$, then $T \in \mathcal{F}$, so that $T^m \in \mathcal{F}$ by Corollary 5.5, and hence $\mathcal{R}(T^m)$ is closed by Corollary 5.2. Recall that $\mathcal{N}(T^m)$ is closed since T^m is bounded. Thus $\mathcal{R}(T^n)$ and $\mathcal{N}(T^m)$ are subspaces of $\mathcal{H}$.

(b)$\Rightarrow$(c). If (b) holds, then, again, $T \in \mathcal{F}$, so that $T^n \in \mathcal{F}$ by Corollary 5.5, and hence $\mathcal{N}(T^m)$ and $\mathcal{N}(T^{m*})$ are finite dimensional, and $\mathcal{R}(T^m)$ is closed by Corollary 5.2. Moreover, suppose there exists $m \in \mathbb{N}$ such that

$$\mathcal{R}(T^m) + \mathcal{N}(T^m) = \mathcal{H} \quad and \quad \mathcal{R}(T^m) \cap \mathcal{N}(T^m) = \{0\}.$$

Since $\mathcal{R}(T^m)$ is closed, $\mathcal{R}(T^m) + \mathcal{R}(T^m)^\perp = \mathcal{H}$ (Projection Theorem), where $\mathcal{R}(T^m)^\perp = \mathcal{N}(T^{m*})$ (Lemma 1.4). Hence,

$$\mathcal{R}(T^m) + \mathcal{N}(T^{m*}) = \mathcal{H} \quad and \quad \mathcal{R}(T^m) \cap \mathcal{N}(T^{m*}) = \{0\}.$$

Thus $\mathcal{N}(T^m)$ and $\mathcal{N}(T^{m*})$ are both algebraic complements of $\mathcal{R}(T^m)$, and so they have the same (finite) dimension (which, in fact, is the codimension of $\mathcal{R}(T^m)$ — see, e.g., [66, Theorem 2.18]). Therefore, $\mathrm{ind}\,(T^m) = 0$. Since $T \in \mathcal{F}$, it follows that $m \,\mathrm{ind}\,(T) = \mathrm{ind}\,(T^m) = 0$ (see Corollary 5.5). Thus (recalling that $m \neq 0$), $\mathrm{ind}\,(T) = 0$, so that $T \in \mathcal{W}$. □

Therefore, since $\mathcal{G}[\mathcal{H}] \subset \mathcal{W}$ (cf. Remark 5.7(b)), it follows that

$$\mathcal{G}[\mathcal{H}] \subset \mathcal{B} \subset \mathcal{W} \subset \mathcal{F}.$$

Note that the inclusion $\mathcal{B} \subset \mathcal{W}$ can be otherwise proved by using Proposition 5.J. Also note that the assertion "T is Browder and not invertible" (i.e., $T \in \mathcal{B}\backslash\mathcal{G}[\mathcal{H}]$) means "$T \in \mathcal{B}$ and $0 \in \sigma(T)$". The following theorem says that if this happens, then 0 must be an isolated point of $\sigma(T)$; precisely, $T \in \mathcal{B}$ and $0 \in \sigma(T)$ *if and only if* $T \in \mathcal{F}$ *and* $0 \in \sigma_{\mathrm{iso}}(T)$.

Theorem 5.34. Take an operator $T \in \mathcal{B}[\mathcal{H}]$.

(a) *If* $T \in \mathcal{B}$ *and* $0 \in \sigma(T)$, *then* $0 \in \pi_0(T)$.

(b) *If* $T \in \mathcal{F}$ *and* $0 \in \sigma_{\mathrm{iso}}(T)$, *then* $T \in \mathcal{B}$.

Proof. (a) Recall: $\sigma_0(T) = \{\lambda \in \sigma(T)\colon \lambda I - T \in \mathcal{W}\}$ and $\mathcal{B} \subset \mathcal{W}$. Therefore, if $T \in \mathcal{B}$ and $0 \in \sigma(T)$, then

$$0 \in \sigma_0(T).$$

Moreover, Theorem 5.33 ensures that there is an integer $m \geq 1$ for which

$$\mathcal{R}(T^m) + \mathcal{N}(T^m) = \mathcal{H} \quad \text{and} \quad \mathcal{R}(T^m) \cap \mathcal{N}(T^m) = \{0\}.$$

Equivalently,

$$\mathcal{R}(T^m) \oplus \mathcal{N}(T^m) = \mathcal{H} \quad \text{and} \quad \mathcal{R}(T^m) \cap \mathcal{N}(T^m) = \{0\}$$

(where the direct sum is not necessarily orthogonal and equality in fact means equivalence; that is, up to a topological isomorphism). Since both $\mathcal{R}(T^m)$ and $\mathcal{N}(T^m)$ are T^m-invariant, it follows that

$$T^m = T^m|_{\mathcal{R}(T^m)} \oplus T^m|_{\mathcal{N}(T^m)}.$$

Since $\mathcal{N}(T^m) \neq \{0\}$ (otherwise T^m would be invertible) and $T^m|_{\mathcal{N}(T^m)} = O$ acts on the nonzero space $\mathcal{N}(T^m)$, it follows that $T^m = T^m|_{\mathcal{R}(T^m)} \oplus O$. Thus (see Proposition 2.F even though the direct sum is not orthogonal)

$$\sigma(T)^m = \sigma(T^m) = \sigma(T^m|_{\mathcal{R}(T^m)}) \cup \{0\}$$

by Theorem 2.7 (Spectral Mapping). Since $T^m|_{\mathcal{R}(T^m)}\colon \mathcal{R}(T^m) \to \mathcal{R}(T^m)$ is surjective and injective, and since $\mathcal{R}(T^m)$ is closed because $T^m \in \mathcal{F}$, it follows that $T^m|_{\mathcal{R}(T^m)} \in \mathcal{G}[\mathcal{R}(T^m)]$. In other words, $0 \in \rho(T^m|_{\mathcal{R}(T^m)})$, and therefore, $0 \notin \sigma(T^m|_{\mathcal{R}(T^m)})$. Thus $\sigma(T^m)$ is disconnected, and hence 0 is an isolated point of $\sigma(T)^m = \sigma(T^m)$. Then 0 is an isolated point of $\sigma(T)$. That is,

$$0 \in \sigma_{\mathrm{iso}}(T).$$

Outcome: $0 \in \pi_0(T) = \sigma_{\mathrm{iso}}(T) \cap \sigma_0(T)$.

(b) Take an arbitrary $m \in \mathbb{N}$. If $0 \in \sigma_{\mathrm{iso}}(T)$, then $0 \in \sigma_{\mathrm{iso}}(T^m)$. Indeed, if 0 is an isolated pont of $\sigma(T)$, then $0 \in \sigma(T)^m = \sigma(T^m)$, and hence 0 is an isolated pont of $\sigma(T)^m = \sigma(T^m)$. Therefore,

$$\sigma(T^m) = \{0\} \cup \Delta,$$

where Δ is a compact subset of $\mathbb{C}$ that does not contain 0, which means that $\{0\}$ and Δ are complementary spectral sets for the operator T^m (i.e., they form a disjoint partition for $\sigma(T^m)$). Let $E_0 = E_{\{0\}} \in \mathcal{B}[\mathcal{H}]$ be the Riesz idempotent associated with 0. Corollary 4.22 ensures that

$$\mathcal{N}(T^m) \subseteq \mathcal{R}(E_0).$$

Claim. If $T \in \mathcal{F}$ and $0 \in \sigma_{\mathrm{iso}}(T)$, then $\dim \mathcal{R}(E_0) < \infty$.

Proof. Theorem 5.19.

Thus, if $T \in \mathcal{F}$ and $0 \in \sigma_{\mathrm{iso}}(T)$, then $\dim \mathcal{N}(T^m) \leq \dim \mathcal{R}(E_0) < \infty$. Since $\{\mathcal{N}(T^n)\}$ is a nondecreasing sequence of subspaces, $\{\dim \mathcal{N}(T^n)\}$ is a nondecreasing sequence of positive integers, which is bounded by the (finite) integer $\dim \mathcal{R}(E_0)$. This ensures that

$$T \in \mathcal{F} \ \text{ and } \ 0 \in \sigma_{\mathrm{iso}}(T) \quad \Longrightarrow \quad \mathrm{asc}\,(T) < \infty.$$

Since $\sigma(T)^* = \sigma(T^*)$, we get $0 \in \sigma_{\mathrm{iso}}(T)$ if and only if $0 \in \sigma_{\mathrm{iso}}(T^*)$, and so

$$T \in \mathcal{F} \ \text{ and } \ 0 \in \sigma_{\mathrm{iso}}(T) \quad \Longrightarrow \quad \mathrm{asc}\,(T^*) < \infty,$$

because $T \in \mathcal{F}$ if and only if $T^* \in \mathcal{F}$. Therefore, according to Lemma 5.31, $\mathrm{asc}\,(T) < \infty$ and $\mathrm{dsc}\,(T) < \infty$, which means (by definition) that $T \in \mathcal{B}$.   □

Theorem 5.34 also gives us another characterization of the set $\mathcal{B}$ of all Browder operators from $\mathcal{B}[\mathcal{H}]$ (compare with Corollary 5.26).

Corollary 5.35.

$$\mathcal{B} = \big\{T \in \mathcal{F} \colon 0 \in \rho(T) \cup \sigma_{\mathrm{iso}}(T)\big\} = \big\{T \in \mathcal{F} \colon 0 \in \rho(T) \cup \pi_0(T)\big\}.$$

Proof. Consider the set $\pi_0(T) = \sigma_{\mathrm{iso}}(T) \cap \sigma_0(T)$ of Riesz points of T.

Claim. If $T \in \mathcal{F}$, then $0 \in \sigma_{\mathrm{iso}}(T)$ if and only if $0 \in \pi_0(T)$.

Proof. Theorem 5.19.

Thus the expressions for $\mathcal{B}$ follow from (a) and (b) in Theorem 5.34.   □

Remark 5.36. (a) FINITE DIMENSIONAL. Since $\{\mathcal{N}(T^n)\}$ is nondecreasing and $\{\mathcal{R}(T^n)\}$ is nonincreasing, we get $\dim \mathcal{N}(T^n) \leq \dim \mathcal{N}(T^{n+1}) \leq \dim \mathcal{H}$ and $\dim \mathcal{R}(T^{n+1}) \leq \dim \mathcal{R}(T^n) \leq \dim \mathcal{H}$ for every $n \geq 0$. Thus, if $\mathcal{H}$ is finite dimensional, then $\mathrm{asc}\,(T) < \infty$ and $\mathrm{dsc}\,(T) < \infty$. Therefore (see Remark 5.3(c)),

$$\dim \mathcal{H} < \infty \quad \Longrightarrow \quad \mathcal{B} = \mathcal{B}[\mathcal{H}].$$

(b) COMPLEMENTS. Recall that $\sigma(T) \backslash \sigma_{\mathrm{iso}}(T) = \sigma_{\mathrm{acc}}(T)$, and also that

$$\begin{aligned} \sigma(T) \backslash \pi_0(T) &= \sigma(T) \backslash \big(\sigma_{\mathrm{iso}}(T) \cap \sigma_0(T)\big) \\ &= \big(\sigma(T) \backslash \sigma_{\mathrm{iso}}(T)\big) \cup \big(\sigma(T) \backslash \sigma_0(T)\big) = \sigma_{\mathrm{acc}}(T) \cup \sigma_w(T). \end{aligned}$$

Hence, since $\mathcal{G}[\mathcal{H}] \subset \mathcal{B} \subset \mathcal{W} \subset \mathcal{F}$, we get from Corollary 5.35 that

$$\mathcal{F} \backslash \mathcal{B} = \big\{T \in \mathcal{F} \colon 0 \in \sigma_{\mathrm{acc}}(T)\big\} = \big\{T \in \mathcal{F} \colon 0 \in \big(\sigma_{\mathrm{acc}}(T) \cup \sigma_w(T)\big)\big\}.$$

Since $\mathcal{W} \backslash \mathcal{B} = \mathcal{W} \cap (\mathcal{F} \backslash \mathcal{B})$, it follows that $\mathcal{W} \backslash \mathcal{B} = \big\{T \in \mathcal{W} \colon 0 \in \sigma_{\mathrm{acc}}(T)\big\}$. Moreover, since $\sigma_0(T) = \tau_0(T) \cup \pi_0(T)$, where $\tau_0(T)$ is an open subset of $\mathbb{C}$ (according to Corollary 5.20), and since $\pi(T) = \sigma_{\mathrm{iso}}(T) \cap \sigma_0(T)$, it also follows that $\sigma_0(T) \backslash \sigma_{\mathrm{iso}}(T) = \tau_0(T) \subseteq \sigma_{\mathrm{acc}}(T)$. Thus $\mathcal{W} \backslash \mathcal{B} = \big\{T \in \mathcal{W} \colon 0 \in \tau_0(T)\big\}$ by Corollaries 5.26 and 5.35. Summing up:

$$\mathcal{W}\backslash\mathcal{B} = \big\{T \in \mathcal{W}\colon 0 \in \sigma_{\mathrm{acc}}(T)\big\} = \big\{T \in \mathcal{W}\colon 0 \in \tau_0(T)\big\}.$$

Furthermore, since $\sigma(T)\backslash\sigma_0(T) = \sigma_w(T)$, we get from Corollary 5. 26 that

$$\mathcal{F}\backslash\mathcal{W} = \big\{T \in \mathcal{F}\colon 0 \in \sigma_w(T)\big\}.$$

(c) ANOTHER CHARACTERIZATION. *An operator $T \in \mathcal{B}[\mathcal{H}]$ is Browder if and only if it is Fredholm and $\lambda I - T$ is invertible for sufficiently small $\lambda \neq 0$.*

Indeed, the identity $\mathcal{B} = \{T \in \mathcal{F}\colon 0 \in \rho(T) \cup \sigma_{\mathrm{iso}}(T)\}$ in Corollary 5.35 says that $T \in \mathcal{B}$ if and only if $T \in \mathcal{F}$ and either $0 \in \rho(T)$ or 0 is an isolated point of $\sigma(T) = \mathbb{C}\backslash\rho(T)$. Since $\rho(T)$ is an open subset of $\mathbb{C}$, this means that $T \in \mathcal{B}$ if and only if $T \in \mathcal{F}$ and there exists an $\varepsilon > 0$ such that $B_\varepsilon(0)\backslash\{0\} \subset \rho(T)$, where $B_\varepsilon(0)$ is the open ball centered at 0 with radius ε. This can be restated by saying that $T \in \mathcal{B}$ if and only if $T \in \mathcal{F}$ and $\lambda \in \rho(T)$ whenever $0 \neq |\lambda| < \varepsilon$ for some $\varepsilon > 0$. But this is precisely the claimed assertion.

The *Browder spectrum* of an operator $T \in \mathcal{B}[\mathcal{H}]$ is the set $\sigma_b(T)$ of all complex numbers λ for which $\lambda I - T$ is not Browder,

$$\sigma_b(T) = \big\{\lambda \in \mathbb{C}\colon \lambda I - T \in \mathcal{B}[\mathcal{H}]\backslash\mathcal{B}\big\}$$

(compare with Corollary 5.25). Since an operator lies in $\mathcal{B}$ together with its adjoint, it follows at once that $\lambda \in \sigma_b(T)$ if and only if $\overline{\lambda} \in \sigma_b(T^*)$:

$$\sigma_b(T) = \sigma_b(T^*)^*.$$

Moreover, by the preceding definition, if $\lambda \in \sigma_b(T)$, then $\lambda I - T \notin \mathcal{B}$, so that either $\lambda I - T \notin \mathcal{F}$ or $\lambda I - T \in \mathcal{F}$ and $\mathrm{asc}\,(\lambda I - T) = \mathrm{dsc}\,(\lambda I - T) = \infty$ (cf. the definition of Browder operators). In the former case, $\lambda \in \sigma_e(T) \subseteq \sigma(T)$ according to Corollary 5.12. In the latter case, $\lambda I - T$ is not invertible (because $\mathcal{N}(\lambda I - T) = \{0\}$ if and only if $\mathrm{asc}\,(\lambda I - T) = 0$ — also $\mathcal{R}(\lambda I - T) = \mathcal{H}$ if and only if $\mathrm{dsc}\,(\lambda I - T) = 0$), so that $\lambda \notin \rho(T)$; that is, $\lambda \in \sigma(T)$. Therefore,

$$\sigma_b(T) \subseteq \sigma(T).$$

Since $\mathcal{B} \subset \mathcal{W}$, it also follows by Corollary 5.25 that $\sigma_w(T) \subseteq \sigma_b(T)$. Hence

$$\sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T) \subseteq \sigma(T).$$

Equivalently, $\mathcal{G}[\mathcal{H}] \subset \mathcal{B} \subset \mathcal{W} \subset \mathcal{F}$.

Corollary 5.37. *For every operator $T \in \mathcal{B}[\mathcal{H}]$,*

$$\sigma_b(T) \,=\, \sigma_e(T) \cup \sigma_{\mathrm{acc}}(T) \,=\, \sigma_w(T) \cup \sigma_{\mathrm{acc}}(T).$$

Proof. Recall that $\sigma(T) = \{\lambda\} - \sigma(\lambda I - T)$ by Theorem 2.7 (Spectral Mapping), and so $0 \in \sigma(\lambda I - T)\backslash\sigma_{\mathrm{iso}}(\lambda I - T)$ if and only if $\lambda \in \sigma(T)\backslash\sigma_{\mathrm{iso}}(T)$. Thus, by the definition of $\sigma_b(T)$, and using Corollaries 5.12 and 5.35,

$$\sigma_b(T) = \big\{\lambda \in \mathbb{C}\colon \lambda I - T \notin \mathcal{F} \text{ or } 0 \notin \rho(\lambda I - T) \cup \sigma_{\mathrm{iso}}(\lambda I - T)\big\}$$
$$= \big\{\lambda \in \mathbb{C}\colon \lambda \in \sigma_e(T) \text{ or } 0 \in \sigma(\lambda I - T)\backslash\sigma_{\mathrm{iso}}(\lambda I - T)\big\}$$
$$= \big\{\lambda \in \mathbb{C}\colon \lambda \in \sigma_e(T) \text{ or } \lambda \in \sigma(T)\backslash\sigma_{\mathrm{iso}}(T)\big\}$$
$$= \sigma_e(T) \cup \big(\sigma(T)\backslash\sigma_{\mathrm{iso}}(T)\big) = \sigma_e(T) \cup \sigma_{\mathrm{acc}}(T).$$

However, by Theorem 5.24, $\sigma_w(T) = \sigma_e(T) \cup \kappa(T)$, where $\kappa(T) \subseteq \sigma(T)$ is an open set in $\mathbb{C}$. (Indeed, $\kappa(T) = \bigcup_{k \in \mathbb{Z}\backslash\{0\}} \sigma_k(T)$, which is the union of open sets in $\mathbb{C}$ by Theorem 5.16.) Thus $\kappa(T)\backslash\sigma_{\mathrm{acc}}(T) = \varnothing$, so that

$$\sigma_w(T)\backslash\sigma_{\mathrm{acc}}(T) = \big(\sigma_e(T) \cup \kappa(T)\big)\backslash\sigma_{\mathrm{acc}}(T)$$
$$= \big(\sigma_e(T)\backslash\sigma_{\mathrm{acc}}(T)\big) \cup \big(\kappa(T)\backslash\sigma_{\mathrm{acc}}(T)\big) = \sigma_e(T)\backslash\sigma_{\mathrm{acc}}(T). \quad \square$$

The next result is particularly important: it sets another partition of $\sigma(T)$.

Corollary 5.38. *For every operator* $T \in \mathcal{B}[\mathcal{H}]$,

$$\sigma_b(T) \;=\; \sigma(T)\backslash\pi_0(T).$$

Proof. By the Spectral Picture (Corollary 5.18) and Corollary 5.37 we get

$$\sigma(T)\backslash\sigma_b(T) = \sigma(T)\backslash\big(\sigma_e(T) \cup \sigma_{\mathrm{acc}}(T)\big) = \big(\sigma(T)\backslash\sigma_e(T)\big) \cap \big(\sigma(T)\backslash\sigma_{\mathrm{acc}}(T)\big)$$
$$= \Big(\bigcup_{k \in \mathbb{Z}}\sigma_k(T)\Big) \cap \sigma_{\mathrm{iso}}(T) = \sigma_0(T) \cap \sigma_{\mathrm{iso}}(T) = \pi_0(T),$$

according to the definition of $\pi_0(T)$, since $\sigma_k(T)$ is open in $\mathbb{C}$ for every $0 \neq k \in \mathbb{Z}$ (Theorem 5.16) and so it has no isolated point. $\quad \square$

Since $\sigma_b(T) \cup \pi_0(T) \subseteq \sigma(T)$, it follows by Corollary 5.38 that

$$\sigma_b(T) \cup \pi_0(T) = \sigma(T) \quad \text{and} \quad \sigma_b(T) \cap \pi_0(T) = \varnothing,$$

so that $\{\sigma_b(T), \pi_0(T)\}$ forms a partition of the spectrum $\sigma(T)$. Thus

$$\pi_0(T) = \sigma(T)\backslash\sigma_b(T) = \big\{\lambda \in \sigma(T)\colon \lambda I - T \in \mathcal{B}\big\},$$

and
$$\sigma_b(T) = \sigma(T) \iff \pi_0(T) = \varnothing.$$

Corollary 5.39. *For every operator* $T \in \mathcal{B}[\mathcal{H}]$,

$$\pi_0(T) = \sigma_{\mathrm{iso}}(T)\backslash\sigma_b(T) = \sigma_{\mathrm{iso}}(T)\backslash\sigma_w(T) = \sigma_{\mathrm{iso}}(T)\backslash\sigma_e(T)$$
$$= \pi_{00}(T)\backslash\sigma_b(T) = \pi_{00}(T)\backslash\sigma_w(T) = \pi_{00}(T)\backslash\sigma_e(T).$$

Proof. The identities involving $\sigma_b(T)$ are readily verified. In fact, recall that $\pi_0(T) = \sigma_{\mathrm{iso}}(T) \cap \sigma_0(T) \subseteq \sigma_{\mathrm{iso}}(T) \cap \sigma_{PF}(T) = \pi_{00}(T) \subseteq \sigma_{\mathrm{iso}}(T) \subseteq \sigma(T)$. Since $\sigma_b(T) \cap \pi_0(T) = \varnothing$ and $\sigma(T)\backslash\pi_0(T) = \sigma_b(T)$ (Corollary 5.38), we get

$$\pi_0(T) = \pi_0(T)\backslash\sigma_b(T) \subseteq \pi_{00}(T)\backslash\sigma_b(T) \subseteq \sigma_{\mathrm{iso}}(T)\backslash\sigma_b(T) \subseteq \sigma(T)\backslash\sigma_b(T) = \pi_0(T).$$

To prove the identities involving $\sigma_w(T)$ and $\sigma_e(T)$, proceed as follows. Since $\sigma_0(T) = \sigma(T) \backslash \sigma_w(T)$ (Theorem 5.24), we get

$$\pi_0(T) = \sigma_{\mathrm{iso}}(T) \cap \sigma_0(T) = \sigma_{\mathrm{iso}}(T) \cap \big(\sigma(T) \backslash \sigma_w(T)\big) = \sigma_{\mathrm{iso}}(T) \backslash \sigma_w(T).$$

Moreover, using Theorem 5.24 again, $\sigma_w(T) = \sigma_e(T) \cup \kappa(T)$, where $\kappa(T) = \bigcup_{k \in \mathbb{Z} \backslash \{0\}} \sigma_k(T)$ is the collection of all holes of $\sigma_e(T)$, which is a subset of $\sigma(T)$ that is open in $\mathbb{C}$ (union of open sets — Theorem 5.16). Thus

$$\sigma_{\mathrm{iso}}(T) \backslash \sigma_w(T) = \sigma_{\mathrm{iso}}(T) \backslash (\sigma_e(T) \cup \kappa(T)) = (\sigma_{\mathrm{iso}}(T) \backslash \sigma_e(T)) \cap (\sigma_{\mathrm{iso}}(T) \backslash \kappa(T)).$$

But $\sigma_{\mathrm{iso}}(T) \cap \kappa(T) = \varnothing$ because $\kappa(T)$ is a subset of $\sigma(T)$ that is open in $\mathbb{C}$. Hence $\sigma_{\mathrm{iso}}(T) \backslash \kappa(T) = \sigma_{\mathrm{iso}}(T)$, and so

$$\sigma_{\mathrm{iso}}(T) \backslash \sigma_w(T) = \sigma_{\mathrm{iso}}(T) \backslash \sigma_e(T).$$

Finally, recall again that $\pi_0(T) \subseteq \pi_{00}(T) \subseteq \sigma_{\mathrm{iso}}(T)$. Also, By Theorem 5.24, $\pi_0(T) \subseteq \sigma_0(T) = \sigma(T) \backslash \sigma_w(T)$ and $\sigma_e(T) \subseteq \sigma_w(T)$ and, as we have just verified, $\sigma_{\mathrm{iso}}(T) \backslash \sigma_w(T) = \pi_0(T)$ and $\sigma_{\mathrm{iso}}(T) \backslash \sigma_e(T) = \sigma_{\mathrm{iso}}(T) \backslash \sigma_w(T)$. Thus,

$$\pi_{00}(T) \backslash \sigma_w(T) \subseteq \pi_{00}(T) \backslash \sigma_e(T) \subseteq \sigma_{\mathrm{iso}}(T) \backslash \sigma_e(T)$$
$$= \sigma_{\mathrm{iso}}(T) \backslash \sigma_w(T) = \pi_0(T) = \pi_0(T) \backslash \sigma_w(T) \subseteq \pi_{00}(T) \backslash \sigma_w(T),$$

and therefore

$$\pi_0(T) = \pi_{00}(T) \backslash \sigma_w(T) = \pi_{00}(T) \backslash \sigma_e(T). \qquad \square$$

Remark 5.40. (a) FINITE DIMENSIONAL. Take any $T \in \mathcal{B}[\mathcal{H}]$. Since every operator on a finite-dimensional space is Browder (cf. Remark 5.36(a)), it follows by the very definition of the Browder spectrum $\sigma_b(T)$ that

$$\dim \mathcal{H} < \infty \quad \implies \quad \sigma_b(T) = \varnothing.$$

The converse holds by Remark 5.27(a), since $\sigma_w(T) \subseteq \sigma_b(T)$:

$$\dim \mathcal{H} = \infty \quad \implies \quad \sigma_b(T) \neq \varnothing.$$

Thus

$$\sigma_b(T) \neq \varnothing \quad \Longleftrightarrow \quad \dim \mathcal{H} = \infty,$$

and $\sigma_b(T) = \sigma(T) \backslash \pi_0(T)$ (cf. Corollary 5.38) is a compact set in $\mathbb{C}$ because $\sigma(T)$ is compact and $\pi_0(T) \subseteq \sigma_{\mathrm{iso}}(T)$. Moreover, by Remark 5.27(b),

$$\dim \mathcal{H} < \infty \quad \implies \quad \sigma_e(T) = \sigma_w(T) = \sigma_b(T) = \varnothing.$$

(b) COMPACT OPERATORS. Since $\sigma_{\mathrm{acc}}(K) \subseteq \{0\}$ if K is compact (cf. Corollary 2.20), it follows by Remark 5.27(d), Corollary 5.37, and item (a) that

$$K \in \mathcal{B}_\infty[\mathcal{H}] \ \text{ and } \ \dim \mathcal{H} = \infty \quad \implies \quad \sigma_e(K) = \sigma_w(K) = \sigma_b(K) = \{0\}.$$

(c) FINITE ALGEBRAIC AND GEOMETRIC MULTIPLICITIES. We saw in Corollary 5.28 that the set of Riesz points $\pi_0(T) = \sigma_{\mathrm{iso}}(T) \cap \sigma_0(T)$ and the set of isolated eigenvalues of finite multiplicity $\pi_{00}(T) = \sigma_{\mathrm{iso}}(T) \cap \sigma_{PF}(T)$ coincide if any of the equivalent assertions of Corollary 5.28 holds true. The previous corollary supplies a collection of conditions equivalent to $\pi_0(T) = \pi_{00}(T)$. Indeed, by Corollary 5.39, the following four assertions are pairwise equivalent.

  (i) $\pi_0(T) = \pi_{00}(T)$.

  (ii) $\sigma_e(T) \cap \pi_{00}(T) = \varnothing$ (i.e., $\sigma_e(T) \cap \sigma_{\mathrm{iso}}(T) \cap \sigma_{PF}(T) = \varnothing$).

  (iii) $\sigma_w(T) \cap \pi_{00}(T) = \varnothing$ (i.e., $\sigma_w(T) \cap \sigma_{\mathrm{iso}}(T) \cap \sigma_{PF}(T) = \varnothing$).

  (iv) $\sigma_b(T) \cap \pi_{00}(T) = \varnothing$ (i.e., $\sigma_b(T) \cap \sigma_{\mathrm{iso}}(T) \cap \sigma_{PF}(T) = \varnothing$).

Take any $T \in \mathcal{B}[\mathcal{H}]$ and consider again the following relations:

$$\sigma_0(T) = \sigma(T)\backslash\sigma_w(T), \quad \pi_0(T) = \sigma_{\mathrm{iso}}(T) \cap \sigma_0(T), \quad \pi_{00}(T) = \sigma_{\mathrm{iso}}(T) \cap \sigma_{PF}(T).$$

We have defined a class of operators for which the equivalent assertions of Corollary 5.28 hold true (namely, operators that satisfy Weyl's Theorem). The following equivalent assertions define another important class of operators. An operator for which any of those equivalent assertions holds is said to satisfy *Browder's Theorem*. This will be discussed later in Section 5.5.

Corollary 5.41. *The assertions below are pairwise equivalent.*

(a) $\sigma_0(T) = \pi_0(T)$.

(b) $\sigma(T)\backslash\pi_0(T) = \sigma_w(T)$.

(c) $\sigma(T) = \sigma_w(T) \cup \pi_{00}(T)$.

(d) $\sigma(T) = \sigma_w(T) \cup \sigma_{\mathrm{iso}}(T)$.

(e) $\sigma_0(T) \subseteq \sigma_{\mathrm{iso}}(T)$.

(f) $\sigma_0(T) \subseteq \pi_{00}(T)$.

(g) $\sigma_{\mathrm{acc}}(T) \subseteq \sigma_w(T)$.

(h) $\sigma_w(T) = \sigma_b(T)$.

Proof. Recall that $\{\sigma_w(T), \sigma_0(T)\}$ forms a partition of $\sigma(T)$, and also that $\pi_0(T) \subseteq \pi_{00}(T) \subseteq \sigma(T)$. Thus (a) and (b) are equivalent and, if (a) holds,

$$\sigma(T) = \sigma_w(T) \cup \sigma_0(T) = \sigma_w(T) \cup \pi_0(T) \subseteq \sigma_w(T) \cup \pi_{00}(T) \subseteq \sigma(T),$$

so that (a) implies (c); and (c) implies (d) because $\pi_{00}(T) \subseteq \sigma_{\mathrm{iso}}(T)$:

$$\sigma(T) = \sigma_w(T) \cup \pi_{00}(T) \subseteq \sigma_w(T) \cup \sigma_{\mathrm{iso}}(T) \subseteq \sigma(T).$$

Since $\sigma_0(T) = \sigma(T)\backslash\sigma_w(T)$, it follows that, if (d) holds, then

$$\sigma_0(T) = \sigma(T)\backslash\sigma_w(T) = \big(\sigma_w(T) \cup \sigma_{\mathrm{iso}}(T)\big)\backslash\sigma_w(T) \subseteq \sigma_{\mathrm{iso}}(T),$$

so that (d) implies (e). Since $\pi_0(T) = \sigma_{\text{iso}}(T) \cap \sigma_0(T)$, it follows that (e) is equivalent to (a). Since $\pi_{00}(T) = \sigma_{\text{iso}}(T) \cap \sigma_{PF}(T)$ and $\sigma_0(T) \subseteq \sigma_{PF}(T)$, it also follows that (e) is equivalent to (f). Since $\sigma_{\text{acc}}(T)$ and $\sigma_{\text{iso}}(T)$ are complements of each other in $\sigma(T)$, and $\sigma_w(T)$ and $\sigma_0(T)$ are complements of each other in $\sigma(T)$, it follows that (e) and (g) are equivalent. Similarly, since $\sigma_w(T)$ and $\sigma_0(T)$ are complements of each other in $\sigma(T)$, and $\sigma_b(T)$ and $\pi_0(T)$ also are complements of each other in $\sigma(T)$ (cf. Theorem 5.24 and Corollary 5.38), it follows that (h) and (a) are equivalent as well. $\square$

The Browder spectrum is not invariant under compact perturbation. In fact, if $T \in \mathcal{B}[\mathcal{H}]$ is such that $\sigma_b(T) \neq \sigma_w(T)$ (i.e., if T is such that any of the equivalent assertions in Corollary 5.41 fails — and so all of them fail), then there exists a compact $K \in \mathcal{B}_\infty[\mathcal{H}]$ such that

$$\sigma_b(T + K) \neq \sigma_b(T).$$

Indeed, if $\sigma_b(T) \neq \sigma_w(T)$, then $\sigma_w(T) \subset \sigma_b(T)$, and so $\sigma_b(T)$ is not invariant under compact perturbation because $\sigma_w(T)$ is the largest part of $\sigma(T)$ that remains unchanged under compact perturbation. However, as we shall see in the forthcoming Theorem 5.44 (whose proof is based on Lemmas 5.42 and 5.43 below), $\sigma_b(T)$ is invariant under perturbation by compact operators that commute with T.

Set $\mathcal{A} = \mathcal{B}[\mathcal{H}]$, where $\mathcal{H}$ is an infinite-dimensional complex Hilbert space. Let $\mathcal{A}'$ be a unital closed subalgebra of the *unital complex Banach algebra $\mathcal{A}$*, thus a unital complex Banach algebra itself. Take an arbitrary operator T in $\mathcal{A}'$. Let $\mathcal{B}_\infty[\mathcal{H}]' = \mathcal{B}_\infty[\mathcal{H}] \cap \mathcal{A}'$ denote the collection of all compact operators from $\mathcal{A}'$, and let $\mathcal{F}'$ denote the class of all Fredholm operators in the unital complex Banach algebra $\mathcal{A}'$. That is (cf. Proposition 5.C),

$$\mathcal{F}' = \big\{ T \in \mathcal{A}' \colon AT - I \text{ and } TA - I \text{ are in } \mathcal{B}_\infty[\mathcal{H}]' \text{ for some } A \in \mathcal{A}' \big\}.$$

Therefore, $\mathcal{F}' \subseteq \mathcal{F} \cap \mathcal{A}'$. The inclusion is proper in general (see, e.g., [10, p. 86]), but it becomes an identity if $\mathcal{A}'$ is a $*$-subalgebra of the C*-algebra $\mathcal{A}$; that is, if the unital closed subalgebra $\mathcal{A}'$ is a $*$-algebra (see, e.g., [10, Theorem A.1.3]). However, if the inclusion becomes an identity, then Corollary 5.12 ensures that the essential spectra in $\mathcal{A}$ and in $\mathcal{A}'$ coincide:

$$\mathcal{F}' = \mathcal{F} \cap \mathcal{A}' \implies \sigma_e'(T) = \sigma_e(T),$$

where

$$\sigma_e'(T) = \{\lambda \in \sigma'(T) \colon \lambda I - T \in \mathcal{A}' \backslash \mathcal{F}'\}$$

stands for the essential spectrum of $T \in \mathcal{A}'$ with respect to $\mathcal{A}'$, with $\sigma'(T)$ denoting the spectrum of $T \in \mathcal{A}'$ with respect to the unital complex Banach algebra $\mathcal{A}'$. Similarly, let $\mathcal{W}'$ denote the class of Weyl operators in $\mathcal{A}'$,

$$\mathcal{W}' = \{T \in \mathcal{F}' \colon \text{ind}(T) = 0\},$$

and let
$$\sigma'_w(T) = \{\lambda \in \sigma'(T) \colon \lambda I - T \in \mathcal{A}' \backslash \mathcal{W}'\}$$

stand for the Weyl spectrum of $T \in \mathcal{A}'$ with respect to $\mathcal{A}'$. If $T \in \mathcal{A}'$, set
$$\sigma'_0(T) = \sigma'(T) \backslash \sigma'_w(T) = \{\lambda \in \sigma'(T) \colon \lambda I - T \in \mathcal{W}'\}.$$

Moreover, let $\mathcal{B}'$ stand for the class of Browder operators in $\mathcal{A}'$,
$$\mathcal{B}' = \{T \in \mathcal{F}' \colon 0 \in \rho'(T) \cup \sigma'_{\mathrm{iso}}(T)\}$$

according to Corollary 5.35, where $\sigma'_{\mathrm{iso}}(T)$ denotes the set of all isolated points of $\sigma'(T)$, and $\rho'(T) = \mathbb{C} \backslash \sigma'(T)$ is the resolvent set of $T \in \mathcal{A}'$ with respect to the unital complex Banach algebra $\mathcal{A}'$. Let
$$\sigma'_b(T) = \{\lambda \in \sigma'(T) \colon \lambda I - T \in \mathcal{A}' \backslash \mathcal{B}'\}$$

be the Browder spectrum of $T \in \mathcal{A}'$ with respect to $\mathcal{A}'$.

Lemma 5.42. *Consider the preceding setup. If $T \in \mathcal{A}'$, then*
$$\sigma'(T) = \sigma(T) \quad \Longrightarrow \quad \sigma'_b(T) = \sigma_b(T).$$

Proof. Take an arbitrary operator T in $\mathcal{A}'$. Suppose
$$\sigma'(T) = \sigma(T),$$

so that $\rho'(T) = \rho(T)$ and $\sigma'_{\mathrm{iso}}(T) = \sigma_{\mathrm{iso}}(T)$. Therefore, if $\lambda \in \sigma'_{\mathrm{iso}}(T)$, then the Riesz idempotent associated with λ,
$$E_\lambda = \tfrac{1}{2\pi i} \int_{\Gamma_\lambda} (\nu I - T)^{-1} \, d\nu,$$

lies in $\mathcal{A}'$ (because $\sigma'_{\mathrm{iso}}(T) = \sigma_{\mathrm{iso}}(T)$, and also $(\nu I - T)^{-1} \in \mathcal{A}'$ whenever $(\nu I - T)^{-1} \in \mathcal{B}[\mathcal{H}]$ since $\rho'(T) = \rho(T)$). Thus Theorem 5.19 ensures that
$$\lambda \in \sigma'_0(T) \quad \Longleftrightarrow \quad \dim \mathcal{R}(E_\lambda) < \infty \quad \Longleftrightarrow \quad \lambda \in \sigma_0(T)$$

whenever $\lambda \in \sigma'_{\mathrm{iso}}(T)$, since $\sigma'_{\mathrm{iso}}(T) = \sigma_{\mathrm{iso}}(T)$. Hence
$$\pi'_0(T) = \sigma'_0(T) \cap \sigma'_{\mathrm{iso}}(T) = \sigma_0(T) \cap \sigma_{\mathrm{iso}}(T) = \pi_0(T).$$

Then, according to Corollary 5.38,
$$\sigma_b(T) = \sigma(T) \backslash \pi_0(T) = \sigma'(T) \backslash \pi'_0(T) = \sigma'_b(T). \qquad \square$$

Let $\{T\}'$ denote the *commutant* of $T \in \mathcal{B}[\mathcal{H}]$ (i.e., the collection of all operators in $\mathcal{B}[\mathcal{H}]$ that commute with T). This is a unital subalgebra of the unital complex Banach algebra $\mathcal{B}[\mathcal{H}]$, which is weakly closed in $\mathcal{B}[\mathcal{H}]$ (see, e.g., [63, Problem 3.7]), and hence uniformly closed (i.e., closed in the operator norm topology of $\mathcal{B}[\mathcal{H}]$). Therefore $\mathcal{A}' = \{T\}'$ is a unital closed subalgebra of the unital complex Banach algebra $\mathcal{A} = \mathcal{B}[\mathcal{H}]$.

Lemma 5.43. *Consider the preceding setup with $\mathcal{A}' = \{T\}'$. If $0 \in \sigma'(T)$ and $T \in \mathcal{W}'$, then $0 \in \sigma'_{\mathrm{iso}}(T)$. In other words,*

$$\mathcal{A}' = \{T\}' \quad \text{and} \quad 0 \in \sigma'_0(T) \quad \Longrightarrow \quad 0 \in \sigma'_{\mathrm{iso}}(T).$$

Proof. Take any $T \in \mathcal{B}[\mathcal{H}]$. Let $\mathcal{A}'$ be a unital closed subalgebra of $\mathcal{A} = \mathcal{B}[\mathcal{H}]$ that includes T. Suppose $0 \in \sigma'_0(T)$, which means that $0 \in \sigma'(T)$ and $T \in \mathcal{W}'$. Since $T \in \mathcal{W}'$, Lemma 5.23(c) says that there exists a compact $K \in \mathcal{B}_\infty[\mathcal{H}]' = \mathcal{B}_\infty[\mathcal{H}] \cap \mathcal{A}'$, actually a finite-rank operator, and an operator $A \in \mathcal{A}'$ such that

$$A(T + K) = (T + K)A = I.$$

Hence $A \in \mathcal{A}'$ is invertible with inverse $A^{-1} = T + K \in \mathcal{A}'$. If $\mathcal{A}' = \{T\}'$, then $AK = KA$, and so $A^{-1}K = KA^{-1}$. Let $\mathcal{A}''$ be the unital closed commutative subalgebra of $\mathcal{A}'$ generated by A, A^{-1}, and K. Since $T = A^{-1} - K$, it follows that $\mathcal{A}''$ includes T. Let $\sigma'_{\mathrm{iso}}(T)$ and $\sigma''_{\mathrm{iso}}(T)$ stand for the sets of isolated points of the spectra $\sigma'(T)$ and $\sigma''(T)$ of T with respect to the Banach algebras $\mathcal{A}'$ and $\mathcal{A}''$, respectively. Since $\mathcal{A}'' \subseteq \mathcal{A}'$, it follows by Proposition 2.Q(a) that $\sigma'(T) \subseteq \sigma''(T)$. Hence $0 \in \sigma''(T)$ because $0 \in \sigma'_0(T) \subseteq \sigma'(T)$.

Claim. $0 \in \sigma''_{\mathrm{iso}}(T)$.

Proof. Let $\widehat{\mathcal{A}}''$ denote the collection of all algebra homomorphisms of $\mathcal{A}''$ into $\mathbb{C}$. Recall from Proposition 2.Q(b) that

$$\sigma''(A^{-1}) = \big\{\Phi(A^{-1}) \in \mathbb{C}\colon \Phi \in \widehat{\mathcal{A}}''\big\},$$

which is bounded away from zero (since $0 \in \rho''(A^{-1})$), and

$$\sigma''(K) = \big\{\Phi(K) \in \mathbb{C}\colon \Phi \in \widehat{\mathcal{A}}''\big\} = \{0\} \cup \big\{\Phi(K) \in \mathbb{C}\colon \Phi \in \widehat{\mathcal{A}}''_F\big\},$$

where $\widehat{\mathcal{A}}''_F \subseteq \widehat{\mathcal{A}}''$ is a set of nonzero homomorphisms, which is *finite* because K is finite rank (and so K has a finite spectrum — Corollary 2.19). Note that $0 \in \sigma''(T) = \sigma''(A^{-1} - K)$ if and only if $0 = \Phi(A^{-1} - K) = \Phi(A^{-1}) - \Phi(K)$ for some $\Phi \in \mathcal{A}''$. If $\Phi \in \mathcal{A}'' \backslash \mathcal{A}''_F$, then $\Phi(K) = 0$ so that $\Phi(A^{-1}) = 0$, which is a contradiction (because $0 \notin \sigma(A^{-1})$). Thus $\Phi \in \mathcal{A}''_F$, and hence $\Phi(A^{-1} - K) = \Phi(A^{-1}) - \Phi(K) = 0$, so that $\Phi(A^{-1}) = \Phi(K)$, for at most a finite number of homomorphisms Φ in $\mathcal{A}''_F$. Therefore, since the set $\{\Phi(A^{-1}) \in \mathbb{C}\colon \Phi \in \widehat{\mathcal{A}}''\} = \sigma''(A^{-1})$ is bounded away from zero, and since the set

$$\big\{\Phi \in \mathcal{A}''_F\colon \Phi(A^{-1}) = \Phi(K)\big\} = \big\{\Phi \in \mathcal{A}''_F\colon 0 \in \sigma''(A^{-1} - K)\big\}$$

is finite, it follows that 0 is an isolated point of $\sigma''(A^{-1} - K) = \sigma''(T)$, which concludes the proof of the claimed result.

Since $0 \in \sigma'(T) \subseteq \sigma''(T)$, it then follows that $0 \in \sigma'_{\mathrm{iso}}(T)$. $\square$

The next characterization of the Browder spectrum is the counterpart of the very definition of the Weyl spectrum. It says that $\sigma_b(T)$ is the largest

part of $\sigma(T)$ that remains unchanged under compact perturbations in the commutant of T. That is, $\sigma_b(T)$ *is the largest part of* $\sigma(T)$ *such that* [69, 59]

$$\sigma_b(T + K) = \sigma_b(T) \quad \text{for every} \quad K \in \mathcal{B}_\infty[\mathcal{H}] \cap \{T\}'.$$

Theorem 5.44. *For every* $T \in \mathcal{B}[\mathcal{H}]$,

$$\sigma_b(T) = \bigcap_{K \in \mathcal{B}_\infty[\mathcal{H}] \cap \{T\}'} \sigma(T + K).$$

Proof. Take any operator $T \in \mathcal{B}[\mathcal{H}]$, and let $\mathcal{A}'$ be a unital closed subalgebra of the unital complex Banach algebra $\mathcal{A} = \mathcal{B}[\mathcal{H}]$ (thus a unital complex Banach algebra itself). Let $\sigma'(T)$, $\sigma'_b(T)$, and $\sigma'_w(T)$ be the spectrum, the Browder spectrum, and the Weyl spectrum of T with respect to $\mathcal{A}'$, respectively.

Claim 1. If $\mathcal{A}' = \{T\}'$, then $\sigma'(T) = \sigma(T)$.

Proof. Suppose $\mathcal{A}' = \{T\}'$. Trivially, $T \in \mathcal{A}'$. Let $\mathcal{P} = \mathcal{P}(T)$ be the collection of all polynomials $p(T)$ in T with complex coefficients, which is a unital commutative subalgebra of $\mathcal{B}[\mathcal{H}]$. Consider the collection $\mathcal{T}$ of all unital commutative subalgebras of $\mathcal{B}[\mathcal{H}]$ containing T. Note that every element of $\mathcal{T}$ is included in $\mathcal{A}'$, and also that $\mathcal{T}$ is partially ordered (in the inclusion ordering) and nonempty (e.g., $\mathcal{P} \in \mathcal{T}$). Moreover, every chain in $\mathcal{T}$ has an upper bound in $\mathcal{T}$ (the union of all subalgebras in a given chain of subalgebras in $\mathcal{T}$ is again a subalgebra in $\mathcal{T}$). Thus Zorn's Lemma says that $\mathcal{T}$ has a maximal element, say $\mathcal{A}'' = \mathcal{A}''(T) \in \mathcal{T}$. Hence there is a maximal commutative subalgebra $\mathcal{A}''$ of $\mathcal{B}[\mathcal{H}]$ containing T (which is unital and closed — see the paragraph that precedes Proposition 2.Q). Therefore,

$$\mathcal{A}'' \subseteq \mathcal{A}' \subseteq \mathcal{A}.$$

Let $\sigma''(T)$ denote the spectrum of T with respect to $\mathcal{A}''$. If the preceding inclusions hold true, and since $\mathcal{A}''$ is a maximal commutative subalgebra of $\mathcal{A}$, then Proposition 2.Q(a,b) ensure that the following relations also hold true:

$$\sigma(T) \subseteq \sigma'(T) \subseteq \sigma''(T) = \sigma(T).$$

Claim 2. If $\mathcal{A}' = \{T\}'$, then $\sigma_b(T) = \sigma'_b(T)$.

Proof. Claim 1 and Lemma 5.42.

Claim 3. If $\mathcal{A}' = \{T\}'$, then $\sigma'_b(T) = \sigma'_w(T)$.

Proof. Recall that $\lambda \in \sigma'_0(T)$ if and only if $\lambda \in \sigma'(T)$ and $\lambda I - T \in \mathcal{W}'$. But $\lambda \in \sigma'(T)$ if and only if $0 \in \sigma'(\lambda I - T)$ by the Spectral Mapping Theorem (Theorem 2.7). Hence,

$$\lambda \in \sigma'_0(T) \quad \Longleftrightarrow \quad 0 \in \sigma'_0(\lambda I - T).$$

Since $\mathcal{A}' = \{T\}' = \{\lambda I - T\}'$ it follows by Lemma 5.43 that

$$0 \in \sigma_0'(\lambda I - T) \quad \Longrightarrow \quad 0 \in \sigma_{\text{iso}}'(\lambda I - T).$$

However, applying the Spectral Mapping Theorem again,

$$0 \in \sigma_{\text{iso}}'(\lambda I - T) \quad \Longleftrightarrow \quad \lambda \in \sigma_{\text{iso}}'(T).$$

Therefore,

$$\sigma_0'(T) \subseteq \sigma_{\text{iso}}'(T),$$

which means that (cf. Corollary 5.41)

$$\sigma_w'(T) = \sigma_b'(T).$$

Claim 4. $\sigma_w'(T) = \bigcap_{K \in \mathcal{B}_\infty[\mathcal{H}] \cap \mathcal{A}'} \sigma(T + K).$

Proof. This is the very definition of the Weyl spectrum of T with respect to $\mathcal{A}'$. That is, $\sigma_w'(T) = \bigcap_{K \in \mathcal{B}_\infty[\mathcal{H}]'} \sigma(T + K)$, where $\mathcal{B}_\infty[\mathcal{H}]' = \mathcal{B}_\infty[\mathcal{H}] \cap \mathcal{A}'$.

By Claims 2, 3, and 4 we get

$$\sigma_b(T) = \bigcap_{K \in \mathcal{B}_\infty[\mathcal{H}] \cap \{T\}'} \sigma(T + K). \qquad \square$$

5.5 Remarks on Browder and Weyl Theorems

This final section consist of a brief survey on Weyl's and Browder's Theorems. As such, and unlike the previous sections, some of the assertions discussed here, instead of being fully proved, will be accompanied by a reference to indicate where the proof can be found.

Take any operator $T \in \mathcal{B}[\mathcal{H}]$, and consider the partitions $\{\sigma_w(T), \sigma_0(T)\}$ and $\{\sigma_b(T), \pi_0(T)\}$ of the spectrum of T in terms of Weyl and Browder spectra $\sigma_w(T)$ and $\sigma_b(T)$ and their complements $\sigma_0(T)$ and $\pi_0(T)$ in $\sigma(T)$ as in Theorem 5.24 and Corollary 5.38, so that

$$\sigma_w(T) = \sigma(T) \backslash \sigma_0(T) \quad \text{and} \quad \sigma_b(T) = \sigma(T) \backslash \pi_0(T),$$

where

$$\sigma_0(T) = \big\{\lambda \in \sigma(T) \colon \lambda I - T \in \mathcal{W}\big\} \quad \text{and} \quad \pi_0(T) = \sigma_{\text{iso}}(T) \cap \sigma_0(T).$$

Although $\sigma_0(T) \subseteq \sigma_{PF}(T)$ and $\sigma_{\text{acc}}(T) \subseteq \sigma_b(T)$ (i.e., $\pi_0(T) \subseteq \sigma_{\text{iso}}(T)$) for every operator T, it happens that, in general, $\sigma_0(T)$ may not be included in $\sigma_{\text{iso}}(T)$ or, equivalently, $\sigma_{\text{acc}}(T)$ may not be included in $\sigma_w(T)$. Recall that

$$\pi_{00}(T) = \sigma_{\text{iso}}(T) \cap \sigma_{PF}(T).$$

Definition 5.45. An operator T is said to *satisfy Weyl's Theorem* (or *Weyl's Theorem holds* for T) if any of the equivalent assertions of Corollary 5.28 holds true. Therefore, *T satisfies Weyl's Theorem if*

$$\sigma_0(T) = \sigma_{\mathrm{iso}}(T) \cap \sigma_{PF}(T),$$

that is, $\sigma_0(T) = \pi_{00}(T)$ or, equivalently, $\sigma(T) \backslash \pi_{00}(T) = \sigma_w(T)$.

Further necessary and sufficient conditions for an operator to satisfy Weyl's Theorem can be found in [44]. See also [58, Chapter 11].

Definition 5.46. An operator T is said to *satisfy Browder's Theorem* (or *Browder's Theorem holds* for T) if any of the equivalent assertions of Corollary 5.41 holds true. Therefore, T *satisfies Browder's Theorem* if

$$\sigma_0(T) \subseteq \sigma_{\mathrm{iso}}(T) \cap \sigma_{PF}(T),$$

that is, $\sigma_0(T) \subseteq \pi_{00}(T)$ or, equivalently,

$$\sigma_0(T) = \pi_0(T), \quad \text{or} \quad \sigma_0(T) \subseteq \sigma_{\mathrm{iso}}(T), \quad \text{or}$$
$$\sigma_{\mathrm{acc}}(T) \subseteq \sigma_w(T), \quad \text{or} \quad \sigma_w(T) = \sigma_b(T).$$

A word on terminology. The expressions "T satisfies Weyl's Theorem" and "T satisfies Browder's Theorem" are the usual terminologies and we shall stick to them, although saying that T "satisfies Weyl's (or Browder's) *property*" rather than "satisfies Weyl's (or Browder's) *Theorem*" would perhaps sound more appropriate.

Remark 5.47. (a) SUFFICIENCY FOR WEYL'S.

$$\sigma_w(T) = \sigma_{\mathrm{acc}}(T) \quad \Longrightarrow \quad T \text{ satisfies Weyl's Theorem,}$$

according to Corollary 5.28.

(b) BROWDER'S NOT WEYL'S. Consider Definitions 5.45 and 5.46. If T satisfies Weyl's Theorem, then it obviously satisfies Browder's Theorem. An operator T satisfies Browder's but not Weyl's Theorem if and only if the proper inclusion $\sigma_0(T) \subset \sigma_{\mathrm{iso}}(T) \cap \sigma_{PF}(T)$ holds true, which implies that there exists an isolated eigenvalue of finite multiplicity not in $\sigma_0(T)$ (i.e., there exists an isolated eigenvalue of finite multiplicity in $\sigma_w(T) = \sigma(T) \backslash \sigma_0(T)$). Outcome:

If T satisfies Browder's Theorem but not Weyl's Theorem, then
$$\sigma_w(T) \cap \sigma_{\mathrm{iso}}(T) \cap \sigma_{PF}(T) \neq \varnothing.$$

(c) EQUIVALENT CONDITION. The preceding result can be extended as follows (cf. Remark 5.40(c)). Consider the equivalent assertions of Corollary 5.28 and of Corollary 5.41. If Browder's Theorem holds and $\pi_0(T) = \pi_{00}(T)$, then Weyl's Theorem holds (i.e., if $\sigma_0(T) = \pi_0(T)$ and $\pi_0(T) = \pi_{00}(T)$, then $\sigma_0(T) = \pi_{00}(T)$ tautologically). Conversely, if Weyl's Theorem holds, then $\pi_0(T) = \pi_{00}(T)$ (see Corollary 5.28), and Browder's Theorem holds trivially. Summing up:

Weyl's Theorem holds $\iff$ *Browder's Theorem holds and $\pi_0(T) = \pi_{00}(T)$.*

In other words, Weyl's Theorem holds if and only if Browder's Theorem and any of the equivalent assertions of Remark 5.40(c) hold true.

(d) TRIVIAL CASES. By Remark 5.27(a), if $\dim \mathcal{H} < \infty$, then $\sigma_0(T) = \sigma(T)$. Thus, according to Corollary 2.19, $\dim \mathcal{H} < \infty \Longrightarrow \sigma_0(T) = \pi_{00}(T) = \sigma(T)$:

>   *Every operator on a finite-dimensional space satisfies Weyl's Theorem.*

This extends to finite-rank but not to compact operators (see examples in [38]). On the other hand, if $\sigma_P(T) = \varnothing$, then $\sigma_0(T) = \pi_{00}(T) = \varnothing$:

>   *Every operator without eigenvalues satisfies Weyl's Theorem.*

Since $\sigma_0(T) \subseteq \sigma_{PF}(T)$, this extends to operators with $\sigma_{PF}(T) = \varnothing$.

Investigating quadratic forms with compact difference, Hermann Weyl proved in 1909 that Weyl's Theorem holds for self-adjoint operators (i.e., *every self-adjoint operator satisfies Weyl's Theorem*) [90]. We shall present a contemporary proof of the original Weyl's result. First recall from Sections 1.5 and 1.6 the following elementary (proper) inclusion of classes of operators:

$$\text{Self-Adjoint} \quad \subset \quad \text{Normal} \quad \subset \quad \text{Hyponormal}.$$

Theorem 5.48. (WEYL'S THEOREM). *If* $T \in \mathcal{B}[\mathcal{H}]$ *is self-adjoint, then*

$$\sigma_0(T) = \sigma_{\mathrm{iso}}(T) \cap \sigma_{PF}(T).$$

Proof. Take any operator $T \in \mathcal{B}[\mathcal{H}]$.

Claim 1. If T is self-adjoint, then $\sigma_0(T) = \pi_0(T)$.

Proof. If T is self-adjoint, then $\sigma(T) \subset \mathbb{R}$ (Proposition 2.A). Thus no subset of $\sigma(T)$ is open in $\mathbb{C}$, and hence $\sigma_0(T) = \tau_0(T) \cup \pi_0(T) = \pi_0(T)$ because $\tau_0(T)$ is open in $\mathbb{C}$ (Corollary 5.20).

Claim 2. If T is hyponormal and $\lambda \in \pi_{00}(T)$, then $\mathcal{R}(\lambda I - T)$ is closed.

Proof. $\lambda \in \pi_{00}(T) = \sigma_{\mathrm{iso}}(T) \cap \sigma_{PF}(T)$ if and only if λ is an isolated point of $\sigma(T)$ and $0 < \dim \mathcal{N}(\lambda I - T) < \infty$. Thus, if $\lambda \in \pi_{00}(T)$ and T is hyponormal, then Proposition 4.L ensures that $\dim \mathcal{R}(E_\lambda) < \infty$, where E_λ is the Riesz idempotent associated with λ. Then $\mathcal{R}(\lambda I - T)$ is closed by Corollary 4.22.

Therefore, by Claim 2, if T is hyponormal, then (cf. Corollary 5.22)

$$\pi_0(T) = \big\{\lambda \in \pi_{00}(T): \mathcal{R}(\lambda I - T) \text{ is closed}\big\} = \pi_{00}(T).$$

Thus, in particular, if T is self-adjoint, then $\pi_0(T) = \pi_{00}(T)$, and therefore, according to Claim 1,

$$\sigma_0(T) = \pi_0(T) = \pi_{00}(T) = \sigma_{\mathrm{iso}}(T) \cap \sigma_{PF}(T). \qquad \square$$

The preceding proof actually says that if T is self-adjoint, then T satisfies Browder's Theorem and, if T is hyponormal, then $\pi_0(T) = \pi_{00}(T)$. Thus if T is self-adjoint, then T satisfies Weyl's Theorem by Remark 5.47(c). Theorem 5.48 was extended to normal operators in [83]. This can be verified by using Proposition 5.E, which says that, if T is normal, then $\sigma(T)\backslash\sigma_e(T) = \pi_{00}(T)$. But, by Corollary 5.18, $\sigma(T)\backslash\sigma_e(T) = \bigcup_{k\in\mathbb{Z}\backslash\{0\}}\sigma_k(T) \cup \sigma_0(T)$, where $\bigcup_{k\in\mathbb{Z}\backslash\{0\}}\sigma_k(T)$ is open in $\mathbb{C}$ (cf. Theorem 5.16), and $\pi_{00}(T) = \sigma_{\mathrm{iso}}(T) \cap \sigma_{PF}(T)$ is closed in $\mathbb{C}$. Thus $\sigma_0(T) = \pi_{00}(T)$. Therefore, *every normal operator satisfies Weyl's Theorem*. Moreover, Theorem 5.48 was further extended to hyponormal operators in [24], and to seminormal operators in [15]. In other words:

> *If T or T^* is hyponormal, then T satisfies Weyl's Theorem.*

Some additional definitions and terminologies will be needed. An operator is *isoloid* if every isolated point of its spectrum is an eigenvalue; that is, T is isoloid if $\sigma_{\mathrm{iso}}(T) \subseteq \sigma_P(T)$. A Hilbert space operator T is said to be *dominant* if $\mathcal{R}(\lambda I - T) \subseteq \mathcal{R}(\overline{\lambda}I - T^*)$ for every $\lambda \in \mathbb{C}$ or, equivalently, if for each $\lambda \in \mathbb{C}$ there is an $M_\lambda \geq 0$ such that $(\lambda I - T)(\lambda I - T)^* \leq M_\lambda(\lambda I - T)^*(\lambda I - T)$ [33]. Therefore, *every hyponormal operator is dominant* (with $M_\lambda = 1$) *and isoloid* (Proposition 4.L). Recall that a subspace $\mathcal{M}$ of a Hilbert space $\mathcal{H}$ is invariant for an operator $T \in \mathcal{B}[\mathcal{H}]$ (or T-invariant) if $T(\mathcal{M}) \subseteq \mathcal{M}$, and reducing if it is invariant for both T and T^*. A *part* of an operator is a restriction of it to an invariant subspace, and a *direct summand* is a restriction of it to a reducing subspace. The main result in [15] reads as follows (see also [16]).

Theorem 5.49. *If each finite-dimensional eigenspace of an operator on a Hilbert space is reducing, and if every direct summand of it is isoloid, then it satisfies Weyl's Theorem.*

This is a fundamental result that includes many of the previous results along this line, and has also been frequently applied to yield further results; mainly through the following corollary.

Corollary 5.50. *If an operator on a Hilbert space is dominant and every direct summand of it is isoloid, then it satisfies Weyl's Theorem.*

Proof. Take any $T \in \mathcal{B}[\mathcal{H}]$. The announced result can be restated as follows. If $\mathcal{R}(\lambda I - T) \subseteq \mathcal{R}(\overline{\lambda}I - T^*)$ *for every* $\lambda \in \mathbb{C}$, *and the restriction* $T|_{\mathcal{M}}$ *of T to each reducing subspace $\mathcal{M}$ is such that* $\sigma_{\mathrm{iso}}(T|_{\mathcal{M}}) \subseteq \sigma_P(T|_{\mathcal{M}})$, *then T satisfies Weyl's Theorem*. To prove it we need the following elementary result, which extends Lemma 1.13(a) from hyponormal to dominant operators. Take an arbitrary scalar $\lambda \in \mathbb{C}$.

Claim. If $\mathcal{R}(\lambda I - T) \subseteq \mathcal{R}(\overline{\lambda}I - T^*)$, then $\mathcal{N}(\lambda I - T) \subseteq \mathcal{N}(\overline{\lambda}I - T^*)$.

Proof. Take any $S \in \mathcal{B}[\mathcal{H}]$. If $\mathcal{R}(S) \subseteq \mathcal{R}(S^*)$, then $\mathcal{R}(S^*)^\perp \subseteq \mathcal{R}(S)^\perp$, which is equivalent to $\mathcal{N}(S) \subseteq \mathcal{N}(S^*)$ by Lemma 1.4, thus proving the claimed result.

If $\mathcal{R}(\lambda I - T) \subseteq \mathcal{R}(\overline{\lambda} I - T^*)$, then $\mathcal{N}(\lambda I - T)$ reduces T by the preceding Claim and Lemma 1.14(b). Therefore, if $\mathcal{R}(\lambda I - T) \subseteq \mathcal{R}(\overline{\lambda} I - T^*)$ for every $\lambda \in \mathbb{C}$, then every eigenspace of T is reducing and so is, in particular, every finite-dimensional eigenspace of T. Thus the stated result is a straightforward consequence of Theorem 5.49. $\square$

Since every hyponormal operator is dominant, every direct summand of a hyponormal operator is again hyponormal (in fact, every part of a hyponormal operator is again hyponormal — Proposition 1.P), and every hyponormal operator is isoloid (Proposition 4.L), it follows that Corollary 5.50 offers another proof that *every hyponormal operator satisfies Weyl's Theorem.*

We need a few more definitions and terminologies. An operator $T \in \mathcal{B}[\mathcal{H}]$ is *paranormal* if $\|Tx\|^2 \leq \|T^2 x\| \, \|x\|$ for every $x \in \mathcal{H}$, and *totally hereditarily normaloid* (THN) if all parts of it are normaloid, as well as the inverse of all invertible parts. (Tautologically, totally hereditarily normaloid operators are normaloid.) Hyponormal operators are paranormal and dominant, but paranormal operators are not necessarily dominant. These classes are related by proper inclusion (see [37, Remark 1] and [38, Proposition 2]):

$$\text{Hyponormal} \quad \subset \quad \text{Paranormal} \quad \subset \quad \text{THN} \quad \subset \quad \big(\text{Normaloid} \cap \text{Isoloid}\big).$$

Weyl's Theorem has been extended to classes of nondominant operators that properly include the hyponormal operators. For instance, it was extended to paranormal operators in [88] and, beyond, to totally hereditarily normaloid operators in [36]. In fact [36, Lemma 2.5]:

<center>

If T is totally hereditarily normaloid, then both

T and T^* satisfy Weyl's Theorem.

</center>

So Weyl's Theorem holds for paranormal operators and their adjoints and, in particular, Weyl's Theorem holds for hyponormal operators and their adjoints.

Let T and S be arbitrary operators acting on Hilbert spaces. First we consider their (orthogonal) direct sum. Proposition 2.F(b) says that the spectrum of a direct sum coincides with the union of the spectra of the summands,

$$\sigma(T \oplus S) = \sigma(T) \cup \sigma(S).$$

For the Weyl spectra, only inclusion is ensured. In fact, the Weyl spectrum of a direct sum is included in the union of the Weyl spectra of the summands,

$$\sigma_w(T \oplus S) \subseteq \sigma_w(T) \cup \sigma_w(S),$$

but equality does not hold in general [53]. However, it holds if $\sigma_w(T) \cap \sigma_w(S)$ has empty interior [71],

$$\big(\sigma_w(T) \cap \sigma_w(S)\big)^\circ = \varnothing \quad \Longrightarrow \quad \sigma_w(T \oplus S) = \sigma_w(T) \cup \sigma_w(S).$$

In general, Weyl's Theorem does not transfer from T and S to their direct sum $T \oplus S$. The above identity involving the Weyl spectrum of a direct sum, viz., $\sigma_w(T \oplus S) = \sigma_w(T) \cup \sigma_w(S)$, when it holds, plays an important role in establishing sufficient conditions for a direct sum to satisfy Weyl's Theorem. This was recently investigated in [70] and [38]. As for the problem of transferring Browder's Theorem from T and S to their direct sum $T \oplus S$, the following necessary and sufficient condition was proved in [53, Theorem 4].

Theorem 5.51. *If both operators T and S satisfy Browder's Theorem, then the direct sum $T \oplus S$ satisfies Browder's Theorem if and only if*

$$\sigma_w(T \oplus S) = \sigma_w(T) \cup \sigma_w(S).$$

Now consider the tensor product $T \otimes S$ of a pair of Hilbert space operators T and S (for an expository paper on tensor products that will suffice for our needs see [64]). It is known from [20] that the spectrum of a tensor product coincides with the product of the spectra of the factors,

$$\sigma(T \otimes S) = \sigma(T) \cdot \sigma(S).$$

For the Weyl spectrum it was proved in [57] that the inclusion

$$\sigma_w(T \otimes S) \subseteq \sigma_w(T) \cdot \sigma(S) \ \cup \ \sigma(T) \cdot \sigma_w(S)$$

holds. However, it remained an open question whether the preceding inclusion might be an identity; that is, it was not known if there existed a pair of operators T and S for which the above inclusion was proper. This question was solved quite recently, as we shall see later. Sufficient conditions ensuring that the equality holds were investigated in [67]. For instance, if

$$\sigma_e(T)\backslash\{0\} = \sigma_w(T)\backslash\{0\} \quad \text{and} \quad \sigma_e(S)\backslash\{0\} = \sigma_w(S)\backslash\{0\}$$

(which holds, in particular, for compact operators T and S), or if

$$\sigma_w(T \otimes S) = \sigma_b(T \otimes S)$$

(the tensor product satisfies Browder's Theorem), then [67, Proposition 6]

$$\sigma_w(T \otimes S) = \sigma_w(T) \cdot \sigma(S) \ \cup \ \sigma(T) \cdot \sigma_w(S).$$

Again, Weyl's Theorem does not necessarily transfer from T and S to their tensor product $T \otimes S$. The preceding identity involving the Weyl spectrum of a tensor product, namely, $\sigma_w(T \otimes S) = \sigma_w(T) \cdot \sigma(S) \ \cup \ \sigma(T) \cdot \sigma_w(S)$, when it holds, plays a crucial role in establishing sufficient conditions for a tensor product to satisfy Weyl's Theorem, as was recently investigated in [84], [67], and [68]. As for the problem of transferring Browder's Theorem from T and S to their tensor product $T \otimes S$, the following necessary and sufficient condition was proved in [67, Corollary 6].

Theorem 5.52. *If both operators T and S satisfy Browder's Theorem, then the tensor product $T \otimes S$ satisfies Browder's Theorem if and only if*

$$\sigma_w(T \otimes S) = \sigma_w(T) \cdot \sigma(S) \cup \sigma(T) \cdot \sigma_w(S).$$

According to Theorem 5.52, if there exist operators T and S that satisfy Browder's Theorem, but $T \otimes S$ does not satisfy Browder's Theorem, then the Weyl spectrum identity, viz., $\sigma_w(T \otimes S) = \sigma_w(T) \cdot \sigma(S) \cup \sigma(T) \cdot \sigma_w(S)$, does *not* hold for such a pair of operators. An example of a pair of operators that satisfy Weyl's Theorem (and thus satisfy Browder's Theorem) but whose tensor product does not satisfy Browder's Theorem was recently supplied in [61]. Therefore, [61] and [67] together ensure that *there exists a pair of operators T and S for which the inclusion*

$$\sigma_w(T \otimes S) \subset \sigma_w(T) \cdot \sigma(S) \cup \sigma(T) \cdot \sigma_w(S)$$

is proper; that is, for which the Weyl spectrum identity fails.

5.6 Additional Propositions

Proposition 5.A. $\mathcal{F}_\ell$ *and* $\mathcal{F}_r$ *are open sets in* $\mathcal{B}[\mathcal{H}]$*, and so are* $\mathcal{SF}$ *and* $\mathcal{F}$*.*

Proposition 5.B. *The mapping* $\mathrm{ind}(\,\cdot\,) \colon \mathcal{SF} \to \overline{\mathbb{Z}}$ *is continuous, where the topology on* $\mathcal{SF}$ *is the topology inherited from* $\mathcal{B}[\mathcal{H}]$*, and the topology on* $\overline{\mathbb{Z}}$ *is the discrete topology.*

Proposition 5.C. *Take* $T \in \mathcal{B}[\mathcal{H}]$*. The following assertions are equivalent.*

(a) $T \in \mathcal{F}$*.*

(b) *There exists* $A \in \mathcal{B}[\mathcal{H}]$ *such that* $I - AT$ *and* $I - TA$ *are compact.*

(c) *There exists* $A \in \mathcal{B}[\mathcal{H}]$ *such that* $I - AT$ *and* $I - TA$ *are finite rank.*

Proposition 5.D. *Take an operator* T *in* $\mathcal{B}[\mathcal{H}]$*. If* $\lambda \in \mathbb{C} \backslash \sigma_{\ell e}(T) \cap \sigma_{re}(T)$*, then there exists an* $\varepsilon > 0$ *such that* $\dim \mathcal{N}(\nu I - T)$ *and* $\dim \mathcal{N}(\overline{\nu} I - T^*)$ *are constant in the punctured disk* $B_\varepsilon(\lambda) \backslash \{0\} = \{\lambda \in \mathbb{C} \colon 0 < |\nu - \lambda| < \varepsilon\}$*.*

Proposition 5.E. *If* $T \in \mathcal{B}[\mathcal{H}]$ *is normal, then* $\sigma_{\ell e}(T) = \sigma_{re}(T) = \sigma_e(T)$ *and*

$$\sigma(T) \backslash \sigma_e(T) = \sigma_{\mathrm{iso}}(T) \cap \sigma_{PF}(T) \qquad (\text{i.e.,}\ \ \sigma(T) \backslash \sigma_e(T) = \pi_{00}).$$

Proposition 5.F. *Let* $\mathbb{D}$ *denote the open unit disk centered at the origin of the complex plane, and let* $\mathbb{T} = \partial \mathbb{D}$ *be the unit circle. If* $S_+ \in \mathcal{B}[\ell_+^2]$ *is a unilateral shift on* $\ell_+^2 = \ell_+^2(\mathbb{C})$*, then*

$$\sigma_{\ell e}(S_+) = \sigma_{re}(S_+) = \sigma_e(S_+) = \sigma_C(S_+) = \partial \sigma(S_+) = \mathbb{T},$$

and $\mathrm{ind}(\lambda I - S_+) = -1$ *if* $|\lambda| < 1$ *(i.e., if* $\lambda \in \mathbb{D} = \sigma(S_+) \backslash \partial \sigma(S_+))$*.*

Proposition 5.G. *If $\varnothing \neq \Omega \subset \mathbb{C}$ is a compact set, $\{\Lambda_k\}_{k\in\overline{\mathbb{Z}}}$ is a collection of open sets included in Ω whose nonempty sets are pairwise disjoint, and $\Delta \subseteq \Omega$ is a discrete set* (i.e., *containing only isolated points*), *then there exists $T \in \mathcal{B}[\mathcal{H}]$ such that $\sigma(T) = \Omega$, $\sigma_k(T) = \Lambda_k$ for each $k \neq 0$, $\sigma_0(T) = \Lambda_0 \cup \Delta$, and for each $\lambda \in \Delta$ there is a positive integer $n_\lambda \in \mathbb{N}$ such that $\dim \mathcal{R}(E_\lambda) = n_\lambda$* (*where E_λ is the Riesz idempotent associated with the isolated point $\lambda \in \Delta$*).

Recall from Remark 5.27(b,d) that if $K \in \mathcal{B}_\infty[\mathcal{H}]$ is compact, then either $\sigma_w(K) = \varnothing$ if $\dim \mathcal{H} < \infty$, or $\sigma_w(K) = \{0\}$ if $\dim \mathcal{H} = \infty$. The next result says that, on an infinite-dimensional *separable* Hilbert space, every compact operator is a commutator. (An operator T is a *commutator* if there are operators A and B such that $T = AB - BA$.)

Proposition 5.H. *If $T \in \mathcal{B}[\mathcal{H}]$ is an operator on an infinite-dimensional separable Hilbert space $\mathcal{H}$, then*

$$0 \in \sigma_w(T) \quad \Longrightarrow \quad T \text{ is a commutator.}$$

Proposition 5.I. *Take any operator T in $\mathcal{B}[\mathcal{H}]$.*

(a) *If $\dim \mathcal{N}(T) < \infty$ or $\dim \mathcal{N}(T^*) < \infty$, then*

$\quad$ (a$_1$) $\mathrm{asc}(T) < \infty \implies \dim \mathcal{N}(T) \leq \dim \mathcal{N}(T^*)$,

$\quad$ (a$_2$) $\mathrm{dsc}(T) < \infty \implies \dim \mathcal{N}(T^*) \leq \dim \mathcal{N}(T)$.

(b) *If $\dim \mathcal{N}(T) = \dim \mathcal{N}(T^*) < \infty$, then $\mathrm{asc}(T) < \infty \iff \mathrm{dsc}(T) < \infty$.*

Proposition 5.J. *Suppose $T \in \mathcal{B}[\mathcal{H}]$ is a Fredholm operator.*

(a) *If $\mathrm{asc}(T) < \infty$ and $\mathrm{dsc}(T) < \infty$, then $\mathrm{ind}(T) = 0$.*

(b) *If $\mathrm{ind}(T) = 0$, then $\mathrm{asc}(T) < \infty$ if and only if $\mathrm{dsc}(T) < \infty$.*

Therefore,

$$\mathcal{B} \subseteq \mathcal{W} \subseteq \{T \in \mathcal{W} : \mathrm{asc}(T) < \infty \iff \mathrm{dsc}(T) < \infty\},$$

$$\mathcal{W} \backslash \mathcal{B} = \{T \in \mathcal{W} : \mathrm{asc}(T) = \mathrm{dsc}(T) = \infty\}.$$

The next result is an extension of the *Fredholm Alternative* of Remark 5.7(b), and also of the ultimate form of it in Remark 5.27(c).

Proposition 5.K. *Take any compact operator K in $\mathcal{B}_\infty[\mathcal{H}]$.*

(a) *If $\lambda \neq 0$, then $\lambda I - K \in \mathcal{B}$.*

(b) $\sigma(K) \backslash \{0\} = \pi_0(K) \backslash \{0\} = \sigma_{\mathrm{iso}}(K) \backslash \{0\}$.

Proposition 5.L. *Take $T \in \mathcal{B}[\mathcal{H}]$. If $\sigma_w(T)$ is simply connected* (*so it has no holes*), *then $T + K$ satisfies Browder's Theorem for every $K \in \mathcal{B}_\infty[\mathcal{H}]$.*

Notes: Propositions 5.A to 5.C are standard results on Fredholm and semi-Fredholm operators. For instance, see [75, Proposition 1.25] or [27, Proposition XI.2.6] for Proposition 5.A, and [75, Proposition 1.17] or [27, Proposition XI.3.13] for Proposition 5.B. For Proposition 5.C, see [6, Remark 3.3.3] or [50, Problem 181]. The locally constant dimension of kernels is considered in Proposition 5.D (see, e.g., [27, Theorem XI.6.7]), and a finer analysis of the spectra of normal operators and of unilateral shifts is discussed in Propositions 5.E and 5.F (see, e.g., [27, Proposition XI.4.6] and [27, Example XI.4.10]). *Every spectral picture is attainable* [25]. This is described in Proposition 5.G — see [27, Proposition XI.6.13]. For Proposition 5.H, see [16, §7]. The results of Proposition 5.I are from [23, p. 57] (also see [32]), and Proposition 5.J is an immediate consequence of Proposition 5.I. Regarding the Fredholm Alternative version of Proposition 5.K, for item (a) see [74, Theorem 1.4.6]; item (b) follows from Corollary 5.35, which goes back to Corollary 2.20. The compact perturbation result of Proposition 5.L is from [9, Theorem 11].

Suggested Reading

Aiena [2]

Arveson [6]

Barnes, Murphy, Smyth, and West [10]

Caradus, Pfaffenberger, and Yood [23]

Conway [27, 29]

Douglas [34]

Halmos [50]

Harte [52]

Istrăţescu [58]

Kato [60]

Müller [73]

Murphy [74]

Pearcy [75]

Schechter [82]

Sunder [85]

Taylor and Lay [87]

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